LAPLACE TRANSFORMS.

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2 Marks.

Definition:- Let a function $f(t)$ be continuous and defined for positive values of 't'. The Laplace transformation of $f(t)$ associates a function s defined by the equation

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\,dt \;,\; t>0 \text{ Provided the integral exists,}$$

where 's' is a parameter which may be real (or) complex.

*** State the conditions under which the Laplace Transform of $f(t)$ exists:**

(i) $f(t)$ should be continuous or piecewise continuous in the given closed interval $[a,b]$, where $a>0$.

(ii) $f(t)$ should be of exponential order.

Def:- Exponential order: A function $f(t)$ is said to be of exponential order 's' if $\lim\limits_{t\to\infty} e^{-st} f(t) = 0$

Examples:-

① Write a function for which Laplace transform does not exist. Explain why?

(i) $f(t) = e^{t^3}$.

Sol:- $\lim\limits_{t\to\infty} e^{-st} e^{t^3} \Rightarrow \lim\limits_{t\to\infty} e^{-st+t^3} = e^{\infty} = \infty$ (not finite)

$\therefore e^{t^3}$ is not of exponential order.

$\because L[e^{t^3}]$ does not exist.

(ii) $L\left[\frac{1}{t}\right]$ does not exist, since $1/t \to \infty$ as $t \to 0$

($\because f(t) = 1/t$ is not Continuous)

(iii) $L[\tan t]$ does not exist, since $\tan t$ is not Piecewise continuous.

$\therefore \tan t$ has infinite no. of discontinuities at $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots\ldots$

IIIly $L[\cot t]$ does not exist at $0, \pm\pi, \pm 2\pi$.

Important formula:-

1. $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} \, dx$

2. $\Gamma(1/2) = \sqrt{\pi}$

3. $\Gamma(n+1) = n!$, if n is a +ve integer.

4. $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$

5. $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$

6. $\sin^3\theta = \frac{1}{4}\left[3\sin\theta - \sin 3\theta\right]$

7. $\cos^3\theta = \frac{1}{4}\left[\cos 3\theta + 3\cos\theta\right]$

8. $\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$

9. $\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$

10. $\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$

11. $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

12. $\sin 2A = 2\sin A \cos B$.

13. $\sinh at = \frac{e^{at} - e^{-at}}{2}$

14. $\cosh at = \frac{e^{at} + e^{-at}}{2}$

15. $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

16. $\cos^2\theta = \frac{1+\cos 2\theta}{2}$

Transforms of Elementary Functions:-

1. $L[1] = \frac{1}{s}$, $s>0$

2. $L[K] = \frac{K}{s}$, $s>0$

3. $L[t^n] = \begin{cases} \frac{\Gamma(n+1)}{s^{n+1}} & \text{where 'n' is not an integer } \& \ n>-1 \\ \frac{n!}{s^{n+1}} & \text{where 'n' is a +ve integer.} \end{cases}$

4. $L[e^{at}] = \frac{1}{s-a}$

5. $L[e^{-at}] = \frac{1}{s+a}$

6. $L[a^t] = \frac{1}{s-\log a}$

7. $L[\sin at] = \frac{a}{s^2+a^2}$

8. $L[\cos at] = \frac{s}{s^2+a^2}$

9. $L[\sinh at] = \frac{a}{s^2-a^2}$

10. $L[\cosh at] = \frac{s}{s^2-a^2}$

11. **Linearity Property:**

$$L[af(t) \pm bg(t)] = aL[f(t)] \pm bL[g(t)].$$

RESULTS:-

1. $L[K] = \frac{K}{s}$, $s>0$.

Sol:- $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$$\therefore L[K] = \int_0^\infty e^{-st} K\,dt = K\left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= -\frac{K}{s}\left[e^{-st}\right]_0^\infty \Rightarrow -K/s\left[e^{-\infty} - e^{-0}\right]$$

$$\Rightarrow -\frac{K}{s}[0-1] \qquad \left(\because e^{-\infty} = 0,\ e^0 = 1\right)$$

$$\Rightarrow \frac{K}{s}.$$

2. $L[t^n] = \begin{cases} \dfrac{\Gamma(n+1)}{s^{n+1}} & \text{where 'n' is not an integer \& } n>-1 \\ \dfrac{n!}{s^{n+1}} & \text{where 'n' is a +ve integer.} \end{cases}$

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Sol:- W.K.T $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[t^n] = \int_0^\infty e^{-st} t^n\, dt \quad \text{— (1)}$$

Put $st = x$ | $t \to 0 \Rightarrow x \to 0$

$s\,dt = dx$ | $t \to \infty \Rightarrow x \to \infty$

$dt = dx/s$

$$\therefore (1) \Rightarrow L[t^n] = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s}$$

$$\Rightarrow L[t^n] = \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n\, dx$$

$$\Rightarrow L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}} \quad \left(\because \Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx\right)$$

(n is not an integer).

Where 'n' is a +ve integer, we get $\Gamma(n+1) = n\Gamma(n) = n!$

$$\therefore L[t^n] = \frac{n!}{s^{n+1}}, \quad n \text{ is a +ve integer}$$

3. (i) $L[e^{at}] = \dfrac{1}{s-a}$

Sol:- $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[e^{at}] = \int_0^\infty e^{-st} e^{at}\, dt$$

$$= \int_0^\infty e^{-st+at}\, dt$$

$$= \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$= \frac{1}{-(s-a)} \left[e^{-(s-a)t}\right]_0^\infty$$

$$\Rightarrow \frac{1}{-(s-a)} \left[e^{-\infty} - e^{0}\right]$$

$$\Rightarrow \frac{1}{-(s-a)} [0-1]$$

$$\therefore L[e^{at}] = \frac{1}{s-a}, \quad \text{where } s>a$$

(ii) $L[e^{-at}] = \dfrac{1}{s+a}$

Practice as above Method.

4. (i) $L[\sin at] = \dfrac{a}{s^2+a^2}$ $(s>0)$

Sol:-

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

Formula:

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$$

$$L[\sin at] = \int_0^\infty e^{-st} \sin at\, dt$$

$a = -s$, $b = a$

$$= \left[\frac{e^{-st}}{(-s)^2+a^2}(-s\sin at - a\cos at)\right]_{t=0}^{\infty}$$

$$= \left[\frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at)\right]_{t=0}^{\infty}$$

$(e^{-\infty} = 0$, $e^{0} = 1$

$$= \left[0 - \frac{e^0}{s^2+a^2}(-s\sin 0 - a\cos 0)\right]$$

$$\Rightarrow -\frac{1}{s^2+a^2}(0-a) \Rightarrow \frac{a}{s^2+a^2}, \quad s>0.$$

(ii) $L[\cos at] = \dfrac{s}{s^2+a^2}$ $(s>0)$.

Formula:

$$\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$a = -s$ $b = a$

Sol:-

$$L[\cos at] = \int_0^\infty e^{-st}\cos at\, dt$$

$$= \left[\frac{e^{-st}}{s^2+a^2}(-s\cos at + a\sin at)\right]_0^\infty$$

$$= 0 - \left[\frac{e^0}{s^2+a^2}(-s+0)\right]$$

$$= \frac{s}{s^2+a^2}, \quad s>0.$$

5. (i) $L[\sinh at]$

Sol:-

$$L[\sinh at] = L\left[\frac{e^{at}-e^{-at}}{2}\right]$$

$$= \frac{1}{2}\left\{L[e^{at}] - L[e^{-at}]\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\} \Rightarrow \frac{1}{2}\left\{\frac{s+a-s+a}{(s-a)(s+a)}\right\}$$

$$\Rightarrow \frac{1}{2}\left\{\frac{2a}{s^2-a^2}\right\} \Rightarrow \frac{a}{s^2-a^2}$$

$(a+b)(a-b) = a^2-b^2$

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(ii) $L[\cosh at] = \dfrac{s}{s^2-a^2}$

Sol:- $L[\cosh at] = L\left[\dfrac{e^{at}+e^{-at}}{2}\right]$

$$= \frac{1}{2}\left\{L[e^{at}] + L[e^{-at}]\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} + \frac{1}{s+a}\right\} \Rightarrow \frac{1}{2}\left\{\frac{(s+a)+(s-a)}{(s-a)(s+a)}\right\}$$

$$\Rightarrow \frac{1}{2}\left\{\frac{2s}{s^2-a^2}\right\}$$

$$\Rightarrow \frac{s}{s^2-a^2}$$

6. Linearity Property:-

If a, b are constants & f, g are functions of t,

then $L[af(t) \pm bg(t)] = aL[f(t)] \pm bL[g(t)]$

Sol:- W.K.T $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[af(t) \pm bg(t)] = \int_0^\infty e^{-st}[af(t) \pm bg(t)]\, dt$$

$$= \int_0^\infty e^{-st} af(t)\, dt \pm \int_0^\infty e^{-st}\, b\, g(t)\, dt$$

$$= a\int_0^\infty e^{-st} f(t)\, dt \pm b\int_0^\infty e^{-st} g(t)\, dt$$

$$= aL[f(t)] \pm bL[g(t)].$$

PROBLEMS:-

① $L[t^2+e^{-5t}+8+\sinh 5t]$

Sol:- $L[t^2]+L[e^{-5t}]+L[8]+L[\sinh 5t]$

$$=\frac{2!}{s^3}+\frac{1}{s+5}+\frac{8}{s}+\frac{5}{s^2-25}$$

②. $L[\cos(at+b)]$

Sol:-

$\cos(A+B)=\cos A\cos B-\sin A\sin B$

$$L[\cos(at+b)]=L[\cos at\cos b-\sin at\sin b]$$

$$=\cos b\,L[\cos at]-\sin b\,L[\sin at]$$

$$=\cos b\left[\frac{s}{s^2+a^2}\right]-\sin b\left[\frac{a}{s^2+a^2}\right]$$

$$=\frac{s\cos b-a\sin b}{s^2+a^2}.$$

③ $L[t^{3/2}]$

$n=3/2 \rightarrow$ not an integer

Sol:-

$\Gamma(n+1)=n\Gamma(n)$

$$L[t^n]=\frac{\Gamma(n+1)}{s^{n+1}}$$

$$L[t^{3/2}]=\frac{\Gamma(3/2+1)}{s^{3/2+1}}=\frac{\frac{3}{2}\Gamma(3/2)}{s^{5/2}}=\frac{\frac{3}{2}\Gamma(1/2+1)}{s^{5/2}}$$

$$=\frac{3}{2s^{5/2}}\,\frac{1}{2}\Gamma(1/2)$$

$$=\frac{3\sqrt{\pi}}{4s^{5/2}}.$$

④ $L[e^{3t+5}]$

Sol:

$$L[e^{3t+5}]=L[e^{3t}e^5]$$

$$=e^5L[e^{3t}]\Rightarrow e^5\left(\frac{1}{s-3}\right)$$

$$=\frac{e^5}{s-3}$$

(5) $L[2^t]$

$$L[a^t] = L[e^{\log a^t}] = \frac{1}{s-\log a}$$

Sol:- $L[2^t] = L[e^{\log 2^t}]$

$$= L[e^{t(\log 2)}]$$

$$= \frac{1}{s-\log 2}.$$

(6) $L[\cos^4 t]$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

Sol:- $\cos^4 t = (\cos^2 t)^2 = \left[\frac{1+\cos 2t}{2}\right]^2$

$$= \frac{1}{4}\left[1+2\cos 2t+\cos^2 2t\right]$$

$$= \frac{1}{4}\left[1+2\cos 2t+\frac{1+\cos 4t}{2}\right]$$

$$= \frac{1}{8}\left[2+2\cos 2t+1+\cos 4t\right]$$

$$\cos^4 t = \frac{1}{8}\left[3+4\cos 2t+\cos 4t\right]$$

$$\therefore L[\cos^4 t] = \frac{1}{8}\left\{L[3]+4L[\cos 2t]+L[\cos 4t]\right\}$$

$$= \frac{1}{8}\left\{\frac{3}{s}+4\left(\frac{s}{s^2+4}\right)+\frac{s}{s^2+16}\right\}$$

7. $L[\sin^2 t\cos^3 t]$.

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\cos^3\theta = \frac{\cos 3\theta+3\cos\theta}{4}$$

Sol:- $\sin^2 t\cos^3 t = \left(\frac{1-\cos 2t}{2}\right)\left(\frac{\cos 3t+3\cos t}{4}\right)$

$$= \frac{1}{8}\left\{\cos 3t+3\cos t-\underbrace{\cos 2t\cos 3t}_{\cos A\cos B}-3\cos t\cos 2t\right\}$$

$\cos A\cos B = \frac{1}{2}\left\{\cos(A+B)+\cos(A-B)\right\}$

$$= \frac{1}{8}\left\{\cos 3t+3\cos t-\frac{1}{2}(\cos 5t+\cos t)-\frac{3}{2}(\cos 3t+\cos t)\right\}$$

$$= \frac{1}{8}\left\{\cos 3t+3\cos t-\frac{1}{2}\cos 5t-\frac{1}{2}\cos t-\frac{3}{2}\cos 3t-\frac{3}{2}\cos t\right\}$$

$$= \frac{1}{8}\left\{-\frac{1}{2}\cos 3t - \frac{1}{2}\cos 5t + 3\cos t - 2\cos t\right\}$$

$$= \frac{1}{8}\left\{-\frac{1}{2}\cos 3t - \frac{1}{2}\cos 5t + \cos t\right\}$$

$$= \frac{1}{16}\left\{-\cos 3t - \cos 5t + 2\cos t\right\}$$

$$\sin^2 t\cos^3 t = \frac{1}{16}\left\{2\cos t - \cos 3t - \cos 5t\right\}$$

$$\therefore L[\sin^2 t\cos^3 t] = \frac{1}{16}\left\{2L[\cos t] - L[\cos 3t] - L[\cos 5t]\right\}$$

$$= \frac{1}{16}\left\{\frac{2s}{s^2+1} - \frac{s}{s^2+9} - \frac{s}{s^2+25}\right\}$$

(8). $L[\sin t \sin 2t \sin 3t]$

Sol:-

* $\sin A\cos B = \frac{1}{2}\{\sin(A+B) + \sin(A-B)\}$
* $\sin A\sin B = \frac{1}{2}\{\cos(A-B) - \cos(A+B)\}$
* $\sin 2A = 2\sin A\cos B.$
* $\cos(-\theta) = \cos\theta.$

$\sin t \sin 2t \sin 3t$

$$= \sin t\left\{\frac{\cos(-t) - \cos(5t)}{2}\right\}$$

$$= \sin t\left\{\frac{\cos t - \cos 5t}{2}\right\}$$

$$= \frac{1}{2}\sin t\cos t - \frac{1}{2}\sin t\cos 5t$$

$\downarrow \sin 2A$ $\qquad \downarrow \sin A\cos B$

$$= \frac{1}{4}[\sin 2t] - \frac{1}{2}\left[\frac{\sin 6t + \sin 4t}{2}\right] \Rightarrow \frac{1}{4}\sin 2t - \frac{1}{4}[\sin 6t + \sin 4t]$$

$$\Rightarrow \frac{1}{4}\sin 2t - \frac{1}{4}\sin 6t - \frac{1}{4}\sin 4t$$

$$\therefore L[\sin t\sin 2t\sin 3t] = \frac{1}{4}L[\sin 2t] - \frac{1}{4}L[\sin 6t] - \frac{1}{4}L[\sin 4t]$$

$$= \frac{1}{4}\left\{\frac{2}{s^2+4} - \frac{6}{s^2+36} - \frac{4}{s^2+16}\right\}.$$

9. $L[\cosh^3 t]$

Sol:-

$$\cosh t = \frac{e^t+e^{-t}}{2}$$

$$(a+b)^3 = a^3+b^3+3a^2b+3ab^2.$$

$$(\cosh t)^3 = \left(\frac{e^t+e^{-t}}{2}\right)^3$$

$$= \frac{1}{8}\left[e^{3t}+e^{-3t}+3e^{2t}e^{-t}+3e^{t}e^{-2t}\right]$$

$$\cosh^3 t = \frac{1}{8}\left[e^{3t}+e^{-3t}+3e^{t}+3e^{-t}\right]$$

$$L[\cosh^3 t] = \frac{1}{8}\left\{L[e^{3t}]+L[e^{-3t}]+3L[e^{t}]+3L[e^{-t}]\right\}$$

$$= \frac{1}{8}\left\{\frac{1}{s-3}+\frac{1}{s+3}+\frac{3}{s-1}+\frac{3}{s+1}\right\}.$$

10. Is the linearity Property applicable to $L\left[\frac{1-\cos t}{t}\right]$?

Sol:-

$$L\left[\frac{1-\cos t}{t}\right] = L\left[\frac{1}{t}\right] - L\left[\frac{\cos t}{t}\right]$$

(By linearity Property) Provided the results exists.

$\therefore L\left[\frac{1}{t}\right]$ does not exist. Since $\lim_{t\to 0} 1/t = 1/0 = \infty$

$\therefore L\left[\frac{\cos t}{t}\right]$ does not exist. Since $\lim_{t\to 0}\frac{\cos t}{t} = \frac{\cos 0}{0} \Rightarrow 1/0 = \infty$

$\therefore$ Linearity property is not applicable to $L\left[\frac{1-\cos t}{t}\right]$.

FIRST SHIFTING THEOREM (OR) S-Shifting.

* If $L[f(t)] = F(s)$, then $L[e^{at} f(t)] = F(s-a)$
* If $L[f(t)] = F(s)$, then $L[e^{-at} f(t)] = F(s+a)$

Proof:-

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

$$L[e^{at} f(t)] = \int_0^\infty e^{-st} e^{at} f(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt \Rightarrow F(s-a)$$

$$\text{III}^{ly} \quad L[e^{-at} f(t)] = F(s+a).$$

Note:-

$$* \; L[e^{at} f(t)] = [F(s)]_{s \to s-a} \quad (\text{or}) = \{L[f(t)]\}_{s \to s-a}.$$

$$* \; L[e^{-at} f(t)] = [F(s)]_{s \to s+a}$$

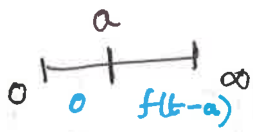
(or)

$$= \{L[f(t)]\}_{s \to s+a}.$$

SECOND SHIFTING THEOREM. [t- shifting]

If $L[f(t)] = F(s)$ & $G(t) = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$ then

$$L[G(t)] = e^{-as} F(s).$$



Sol:-

$$L[G(t)] = \int_0^\infty e^{-st} G(t)\, dt$$

$$= \int_0^a e^{-st} \cdot 0\, dt + \int_a^\infty e^{-st} f(t-a)\, dt$$

$$= \int_{t=a}^{\infty} e^{-st} f(t-a)\, dt \quad \text{—} \; (1)$$

Put $t - a = u$, $dt = du$ | $t = a \Rightarrow u = 0$, $t = \infty \Rightarrow u = \infty$

$$(1) \Rightarrow L[G(t)] = \int_{u=0}^{\infty} e^{-s(u+a)} f(u)\, du$$

$$= \int_0^\infty e^{-su} e^{-as} f(u)\, du$$

$$= e^{-as} \int_0^\infty e^{-su} f(u)\, du$$

'u' is a dummy Variable. Replace 'u' by 't'.

$$= e^{-as} \int_0^\infty e^{-st} f(t)\, dt$$

$$= e^{-as} L[f(t)]$$

(or)

$$= e^{-as} F(s).$$

CHANGE OF SCALE PROPERTY

If $L[f(t)] = F(s)$, then $L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$, $a>0$.

Sol:- W.K.T $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$\therefore L[f(at)] = \int_0^\infty e^{-st} f(at)\,dt$ — (i)

Put $at = x$ | $t=0 \Rightarrow x=0$

$a\,dt = dx$ | $t=\infty \Rightarrow x=\infty$

$dt = dx/a$

$\therefore$ (i) $\Rightarrow L[f(at)] = \int_0^\infty e^{-s\left(\frac{x}{a}\right)} f(x) \frac{dx}{a}$

$= \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)x} f(x)\,dx \Rightarrow \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)t} f(t)\,dt$

Replace dummy variable 'x' by 't'.

$L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$, $a>0$.

PROBLEMS:-

(1). $L\left[\frac{t}{e^t}\right]$

Sol:- $L[t/e^t] = L[te^{-t}]$

$= \{L[t]\}_{s\to s+1}$

$= \left\{\frac{1}{s^2}\right\}_{s\to s+1} \Rightarrow \frac{1}{(s+1)^2}$.

(2). $L[e^{-t}\sin 2t]$.

Sol:- $L[e^{-t}\sin 2t] = \{L[\sin 2t]\}_{s\to s+1}$

$= \left\{\frac{2}{s^2+2^2}\right\}_{s\to s+1}$

$= \frac{2}{(s+1)^2+4} \Rightarrow \frac{2}{s^2+2s+5}$

3. $L[t^2 e^{-2t}]$

Sol:-

$$L[t^2 e^{-2t}] = \{L[t^2]\}_{s \to s+2}$$

$$= \left\{\frac{2}{s^3}\right\}_{s \to s+2} \Rightarrow \frac{2}{(s+2)^3}.$$

4. $L[e^{-3t} \sin t \cos t]$

Sol:-

$$L[e^{-3t} \sin t \cos t] = L\left[e^{-3t} \frac{\sin 2t}{2}\right]$$

$$= \frac{1}{2} L[e^{-3t} \sin 2t]$$

$$= \frac{1}{2}\{L[\sin 2t]\}_{s \to s+3}$$

$$= \frac{1}{2}\left\{\frac{2}{s^2+4}\right\}_{s \to s+3}$$

$$= \frac{1}{2}\left\{\frac{2}{(s+3)^2+4}\right\} \Rightarrow \frac{1}{s^2+13+6s}$$

5. $L[\cosh at \cos at]$

Sol:-

$$L[\cosh at \cos at] = L\left[\left(\frac{e^{at}+e^{-at}}{2}\right)\cos at\right]$$

$$= \frac{1}{2}\left\{L[e^{at}\cos at] + L[e^{-at}\cos at]\right\}$$

$$= \frac{1}{2}\left\{\{L[\cos at]\}_{s \to s-a} + \{L[\cos at]\}_{s \to s+a}\right\}$$

$$= \frac{1}{2}\left\{\left(\frac{s}{s^2+a^2}\right)_{s \to s-a} + \left(\frac{s}{s^2+a^2}\right)_{s \to s+a}\right\}$$

$$= \frac{1}{2}\left\{\frac{s-a}{(s-a)^2+a^2} + \frac{s+a}{(s+a)^2+a^2}\right\}$$

$$= \frac{1}{2}\left\{ \frac{(s-a)\left[(s+a)^2+a^2\right] + (s+a)\left[(s-a)^2+a^2\right]}{\left[(s-a)^2+a^2\right]\left[(s+a)^2+a^2\right]} \right\}$$

$$= \frac{1}{2}\left\{ \frac{(s-a)(s+a)^2 + (s-a)a^2 + (s+a)(s-a)^2 + a^2(s+a)}{(s^2+a^2-2as+a^2)(s^2+a^2+2as+a^2)} \right\}$$

$$= \frac{1}{2}\left\{ \frac{(s-a)(s+a)\left[s+a+s-a\right] + a^2\left[s-a+s+a\right]}{(s^2-2as+2a^2)(s^2+2as+2a^2)} \right\}$$

$$= \frac{1}{2}\left\{ \frac{(s^2-a^2)(2s) + a^2(2s)}{(s^2+2as+2a^2)(s^2-2as+2a^2)} \right\}$$

$$= \frac{1}{2}\left\{ \frac{2s^3-2a^2s+2a^2s}{(s^2+2a^2)^2-4a^2s^2} \right\}$$

$$= \frac{1}{2}\left\{ \frac{2s^3}{s^4+4a^2s^2+4a^4-4a^2s^2} \right\} \Rightarrow \frac{s^3}{s^4+4a^4}.$$

6. $L\left[\dfrac{(\sqrt{t}-1)^2}{\sqrt{t}}\right]$.

Sol:- $L\left[\dfrac{(\sqrt{t}-1)^2}{\sqrt{t}}\right] = L\left[\dfrac{t-2\sqrt{t}+1}{\sqrt{t}}\right]$

$$= L\left[\frac{t}{\sqrt{t}} - \frac{2\sqrt{t}}{\sqrt{t}} + \frac{1}{\sqrt{t}}\right]$$

$$= L\left[t^{1/2} - 2 + t^{-1/2}\right]$$

$$= L\left[t^{1/2}\right] - L[2] + L\left[t^{-1/2}\right]$$

$$= \frac{\frac{1}{2}\Gamma(1/2)}{s^{3/2}} - 2/s + \frac{\Gamma(1/2)}{s^{1/2}}$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} - \frac{2}{s} + \frac{\sqrt{\pi}}{\sqrt{s}} \Rightarrow \frac{\sqrt{\pi}(1+2s)-4\sqrt{s}}{2s^{3/2}} \quad //.$$

$\left(\because L[t^n] = \dfrac{\Gamma(n+1)}{s^{n+1}} \right.$

$L[t^{1/2}] = \dfrac{\Gamma(1/2+1)}{s^{1/2+1}}$

$L[t^{1/2}] = \dfrac{1/2\,\Gamma(1/2)}{s^{3/2}}$

$L[t^{-1/2}] = \dfrac{\Gamma(-1/2+1)}{s^{-1/2+1}}$

$\left. = \dfrac{\Gamma(1/2)}{s^{1/2}} \right)$

7. Find $L[f(t)]$ where $f(t) = \begin{cases} \cos(t - 2\pi/3) & t > 2\pi/3 \\ 0 & t < 2\pi/3 \end{cases}$.

Sol:- W.K.T by Second shifting theorem if

$$L[f(t)] = F(s) \text{ \& } G(t) = \begin{cases} f(t-a) & t>a \\ 0 & t<a \end{cases} \text{ then } L[G(t)] = e^{-as} F(s) \quad \text{—} (*)$$

Here $f(t-a) = \cos(t - 2\pi/3)$

$f(t) = \cos t \qquad a = 2\pi/3$

$$L[f(t)] = F(s) = L[\cos t] = \frac{s}{s^2+1}$$

$$\therefore (*) \Rightarrow L[G(t)] = e^{-2\pi/3}\left[\frac{s}{s^2+1}\right].$$

(OR) ALTIER:-

$$L[G(t)] = \int_0^\infty e^{-st} G(t)\,dt = \int_0^{2\pi/3} e^{-st}\, 0\, dt + \int_{2\pi/3}^{\infty} e^{-st} \cos\left(t - \frac{2\pi}{3}\right) dt$$

$$= \int_{2\pi/3}^{\infty} e^{-st} \cos(t - 2\pi/3)\, dt \quad \text{—} (1)$$

Put $t - \frac{2\pi}{3} = x$ | $t \to 2\pi/3 \Rightarrow x \to 0$

$dt = dx$ | $t \to \infty \Rightarrow x \to \infty$

$$\therefore (1) \Rightarrow \int_0^\infty e^{-s(x+2\pi/3)} \cos x\, dx = \int_0^\infty e^{-sx} e^{-s2\pi/3} \cos x\, dx$$

$$= e^{-2\pi s/3} \int_0^\infty e^{-sx} \cos x\, dx$$

$$= e^{-2\pi s/3} \int_0^\infty e^{-st} \cos t\, dt \qquad (\because x \text{ is a dummy variable})$$

$$= e^{-2\pi s/3}\, L[\cos t]$$

$$= e^{-2\pi s/3}\left[\frac{s}{s^2+1}\right]$$

8. If $L[f(t)] = F(s)$. P.T $L[f(2t)] = \frac{1}{2} F(s/2)$

Sol:- Use the change of scale property theorem (Replace Put $a=2$ & we get

$$L[f(2t)] = \frac{1}{2} F(s/2).$$

TRANSFORMS OF PERIODIC FUNCTIONS.

Periodic :- A function $f(x)$ is said to be "Periodic" if and only if $f(x+P) = f(x)$ is true for some values of P & every value of x.

The Laplace Transformation of a Periodic function $f(t)$ with Period p is given by

$$L[f(t)] = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\, dt.$$

Problems:-

① Find the Laplace transform of the rectangular wave given by $f(t) = \begin{cases} K, & 0 < t < b \\ -K & b < t < 2b. \end{cases}$ with $f(t+2b) = f(t)$.

Sol:- This function is Periodic with Period $2b$. (ie, $p = 2b$)

$$L[f(t)] = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\, dt$$

$$L[f(t)] = \frac{1}{1-e^{-2bs}} \int_0^{2b} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2bs}} \left[\int_0^b e^{-st} K\, dt + \int_b^{2b} e^{-st} (-K)\, dt \right]$$

$$= \frac{K}{1-e^{-2bs}}\left[\int_0^b e^{-st}\,dt - \int_b^{2b} e^{-st}\,dt\right]$$

$$= \frac{K}{1-e^{-2bs}}\left[\left(\frac{e^{-st}}{-s}\right)_0^b - \left(\frac{e^{-st}}{-s}\right)_b^{2b}\right]$$

$$= \frac{K}{1-e^{-2bs}}\left[\left(\frac{e^{-sb}}{-s} - \frac{e^{0}}{-s}\right) - \left(\frac{e^{-2bs}}{-s} - \frac{e^{-bs}}{-s}\right)\right]$$

$$= \frac{K}{1-e^{-2bs}}\left[\frac{1}{s}\left(-e^{-bs}+1+e^{-2bs}-e^{-bs}\right)\right]$$

$$= \frac{K}{s(1-e^{-2bs})}\left[1-2e^{-bs}+e^{-2bs}\right]$$

$$= \frac{K}{s(1-e^{-2bs})}(1-e^{-bs})^2$$

$(\because a^2-b^2=(a+b)(a-b))$

$(1-e^{-2x}) = (1-e^{-x})(1+e^{-x})$

$$= \frac{K}{s\left[(1)^2-(e^{-bs})^2\right]}(1-e^{-bs})^2$$

$$= \frac{K(1-e^{-bs})^2}{s(1-e^{-bs})(1+e^{-bs})}$$

$$\tanh\theta = \frac{1-e^{-2\theta}}{1+e^{-2\theta}}$$

$$= \frac{K}{s}\left[\frac{1-e^{-bs}}{1+e^{-bs}}\right]$$

$\because 2\theta = bs$

$\theta = bs/2$

$$= \frac{K}{s}\tanh\left(\frac{bs}{2}\right).$$

2. Find the Laplace transform of the Half-Sine wave rectifier function $f(t) = \begin{cases} \sin\omega t, & 0 < t < \pi/\omega \\ 0, & \pi/\omega < t < 2\pi/\omega \end{cases}$

Sol:- This function is Periodic with Period, $p = \frac{2\pi}{\omega}$

(18)

$$L[f(t)] = \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \int_0^{2\pi/\omega} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left[\int_0^{\pi/\omega} e^{-st} \sin\omega t\,dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st} \cancel{\phi}\,dt \right]$$

0.

$\int e^{ax} \sin bx\,dx$

$\sin\pi = 0$
$\cos\pi = -1$
$\sin 0 = 0$
$\cos 0 = 1$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left[\frac{e^{-st}}{s^2+\omega^2} (-s\sin\omega t - \omega\cos\omega t) \right]_0^{\pi/\omega}$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left\{ \left[\frac{e^{-s\pi/\omega}}{s^2+\omega^2} \left(-s\sin \cancel{\omega} \tfrac{\pi}{\cancel{\omega}} - \omega\cos \cancel{\omega} \tfrac{\pi}{\cancel{\omega}}\right) \right] - \frac{e^0}{s^2+\omega^2} (-s\sin 0 - \omega\cos 0) \right]$$

upper limit

lower limit

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left\{ \frac{e^{-s\pi/\omega}}{s^2+\omega^2} (0+\omega) - \frac{1}{s^2+\omega^2} (0-\omega) \right\}$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left\{ \frac{\omega e^{-\pi s/\omega}}{s^2+\omega^2} + \frac{\omega}{s^2+\omega^2} \right\}$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left\{ \frac{\omega e^{-\pi s/\omega} + \omega}{s^2+\omega^2} \right\} \Rightarrow \frac{1}{(1+e^{-\pi s/\omega})(1-e^{-\pi s/\omega})} \left\{ \frac{\omega(1+e^{-\pi s/\omega})}{s^2+\omega^2} \right\}$$

$$\Rightarrow \frac{\omega}{(s^2+\omega^2)(1-e^{-\pi s/\omega})}.$$

3. Find the Laplace transform of a square wave function given by $f(t) = \begin{cases} E & \text{for } 0 \le t \le a/2 \\ -E & \text{for } a/2 \le t \le a \end{cases}$ & $f(t+a) = f(t)$.

Sol:- This function is Periodic with Period, $p = a$

$$L[f(t)] = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-as}} \left[\int_0^{a/2} e^{-st}(E)\,dt + \int_{a/2}^{a} e^{-st}(-E)\,dt \right]$$

$$= \frac{E}{1-e^{-as}} \left[\int_0^{a/2} e^{-st}\,dt - \int_{a/2}^{a} e^{-st}\,dt \right]$$

$$= \frac{E}{1-e^{-as}} \left[\left(\frac{e^{-st}}{-s}\right)_0^{a/2} - \left(\frac{e^{-st}}{-s}\right)_{a/2}^{a} \right]$$

$$= \frac{E}{1-e^{-as}} \left\{ \left(\frac{e^{-as/2}}{-s} - \frac{e^{0}}{-s} \right) - \left(\frac{e^{-as}}{-s} - \frac{e^{-as/2}}{-s} \right) \right\}$$

$$= \frac{E}{1-e^{-as}} \left\{ \frac{e^{-as/2}}{-s} + \frac{1}{s} + \frac{e^{-as}}{s} - \frac{e^{-as/2}}{s} \right\}$$

$$= \frac{E}{(1-e^{-as})\,s} \left\{ -e^{-as/2} + 1 + e^{-as} - e^{-as/2} \right\}$$

$$= \frac{E}{(1-e^{-as})\,s} \left\{ 1 - 2e^{-as/2} + e^{-as} \right\}$$

$$= \frac{E}{s(1-e^{-as})} \left(1 - e^{-as/2}\right)^2$$

$$= \frac{E\left(1-e^{-as/2}\right)^2}{s\left[1-e^{-as/2}\right]\left[1+e^{-as/2}\right]}$$

$$\left(\because \tanh\theta = \frac{1-e^{-2\theta}}{1+e^{-2\theta}}\right)$$

$$2\theta = \frac{as}{2}$$

$$\theta = \frac{as}{4}.$$

$$= \frac{E}{s} \left[\frac{1-e^{-as/2}}{1+e^{-as/2}} \right]$$

$$= \frac{E}{s} \tanh\left(\frac{as}{4}\right)$$

4. Find the Laplace Transform of triangular wave function

$$f(t) = \begin{cases} t & 0 < t < a \\ 2a - t & a < t < 2a \end{cases} \quad \text{with } f(t+2a) = f(t).$$

Sol:- This function is Periodic with Period, $p = 2a$.

$$L[f(t)] = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st}\, t\,dt + \int_a^{2a} e^{-st}(2a-t)\,dt \right]$$

Apply Bernoulli's ... Apply Bernoullis

$$= \frac{1}{1-e^{-2as}} \left\{ \left[t\left(\frac{e^{-st}}{-s}\right) - 1\left(\frac{e^{-st}}{s^2}\right) \right]_0^a + \left[(2a-t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{s^2}\right) \right]_a^{2a} \right\}$$

$$= \frac{1}{1-e^{-2as}} \left\{ \left[-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_0^a + \left[-(2a-t)\frac{e^{-st}}{s} + \frac{e^{-st}}{s^2} \right]_a^{2a} \right\}$$

$$= \frac{1}{1-e^{-2as}} \left\{ \left[\underbrace{\left(-\frac{ae^{-as}}{s} - \frac{e^{-as}}{s^2}\right)}_{\text{Upper}} - \underbrace{\left(0 - \frac{e^{0}}{s^2}\right)}_{\text{Lower}} \right] + \left[\underbrace{\left(0 + \frac{e^{-2as}}{s^2}\right)}_{\text{upper}} - \underbrace{\left(-\frac{ae^{-as}}{s} + \frac{e^{-as}}{s^2}\right)}_{\text{Lower}} \right] \right\}$$

$$\Rightarrow \frac{1}{1-e^{-2as}} \left\{ \cancel{-\frac{ae^{-as}}{s}} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + \cancel{\frac{ae^{-as}}{s}} - \frac{e^{-as}}{s^2} \right\}$$

$$\Rightarrow \frac{1}{1-e^{-2as}} \left\{ \frac{1 + e^{-2as} - 2e^{-as}}{s^2} \right\}$$

$$= \frac{1}{(1-e^{-as})(1+e^{-as})} \left[\frac{(1-e^{-as})^2}{s^2} \right]$$

Bernoulli's.

$\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \dots$

$u, u', u'', \dots \rightarrow$ Successive differentiation

$v_1, v_2, v_3 \dots \rightarrow$ Successive Integration

$U \rightarrow$ ILATE (choosen)

$$\Rightarrow \frac{1}{s^2}\left\{\frac{1-e^{-as}}{1+e^{-as}}\right\} \qquad \left(\because \tanh\theta = \frac{1-e^{-2\theta}}{1+e^{-2\theta}}\right.$$

$$\Rightarrow \frac{1}{s^2}\tanh\left(\frac{as}{2}\right) \qquad 2\theta = as$$

$$\theta = as/2.$$

5. Find the Laplace Transform of the function

$$f(t) = \begin{cases} t & 0 < t < \pi/2 \\ \pi - t & \pi/2 < t < \pi \end{cases}$$

Sol:- This function is Periodic with Period, $P = \pi$

$$\because L[f(t)] = \frac{1}{1-e^{-\pi s}}\int_0^{\pi} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-\pi s}}\left\{\int_0^{\pi/2} e^{-st}\, t\, dt + \int_{\pi/2}^{\pi} e^{-st}(\pi - t)\,dt\right\}$$

$$= \frac{1}{1-e^{-\pi s}}\left\{\left[t\left(\frac{e^{-st}}{-s}\right) - 1\cdot\left(\frac{e^{-st}}{s^2}\right)\right]_0^{\pi/2} + \left[(\pi - t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{s^2}\right)\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{1}{1-e^{-\pi s}}\left\{\left[\underbrace{\left(\frac{\pi/2\, e^{-s\pi/2}}{-s} - \frac{e^{-s\pi/2}}{s^2}\right)}_{\text{Upper}} - \underbrace{\left(0 - \frac{e^{0}}{s^2}\right)}_{\text{Lower}}\right] + \left[\underbrace{\left(0 + \frac{e^{-\pi s}}{s^2}\right)}_{\text{Upper}} - \underbrace{\left((\pi - \pi/2)\frac{e^{-\pi s/2}}{-s} + \frac{e^{-\pi s/2}}{s^2}\right)}_{\text{lower}}\right]\right\}$$

$$\Rightarrow \frac{1}{1-e^{-\pi s}}\left\{\cancel{\frac{\pi/2\, e^{-s\pi/2}}{-s}} - \frac{e^{-s\pi/2}}{s^2} + 1/s^2 + \frac{e^{-\pi s}}{s^2} - \cancel{\frac{\pi/2\, e^{-\pi s/2}}{-s}} - \frac{e^{-s\pi/2}}{s^2}\right\}$$

$$\Rightarrow \frac{1}{1-e^{-\pi s}}\left\{-\frac{e^{-s\pi/2}}{s^2} + 1/s^2 + \frac{e^{-\pi s}}{s^2} - \frac{e^{-s\pi/2}}{s^2}\right\}$$

$$\Rightarrow \frac{1}{(1-e^{-\pi s})s^2}\left\{1 + e^{-\pi s} - 2e^{-s\pi/2}\right\} \Rightarrow \frac{1}{(1-e^{-\pi s})s^2}\left\{(1-e^{-s\pi/2})^2\right\}$$

$$\Rightarrow \frac{1}{s^2(1-e^{-\frac{\pi s}{2}})(1+e^{-\pi s/2})}\left\{(1-e^{-s\pi/2})^2\right\}$$

$$\Rightarrow \frac{1}{s^2}\left\{\frac{1-e^{-\pi s/2}}{1+e^{-\pi s/2}}\right\} \Rightarrow \frac{1}{s^2}\tanh\left(\frac{\pi s}{4}\right) //$$

$\left(\because \tanh\theta = \frac{1-e^{-2\theta}}{1+e^{-2\theta}}\right.$

$2\theta = \pi s/2$

$\theta = \pi s/4.$

6. Find the Laplace transform of

$$f(t) = \begin{cases} \cos t & 0 < t < \pi \\ 0 & \pi < t < 2\pi. \end{cases}$$

Sol:- This function is Periodic with Period, $P = 2\pi$

$$L[f(t)] = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2\pi s}}\left\{\int_0^{\pi} e^{-st}\cos t\,dt + \int_{\pi}^{2\pi} \cancel{e^{-st}\,0\,dt}\right\}$$

$a = -s \quad b = 1$

Apply Formula $\int e^{ax}\cos bx\,dx$.

$$= \frac{1}{1-e^{-2\pi s}}\left\{\frac{e^{-st}}{s^2+1}(-s\cos t + \sin t)\right\}_0^{\pi}$$

$$= \frac{1}{1-e^{-2\pi s}}\left\{\frac{e^{-\pi s}}{s^2+1}(-s\cos\pi + \cancel{\sin\pi}) - \frac{e^0}{s^2+1}(-s\cos 0 + \cancel{\sin 0})\right\}$$

$\sin\pi = 0$

$\cos\pi = -1$

$\sin 0 = 0$

$\cos 0 = 1$

$$\Rightarrow \frac{1}{1-e^{-2\pi s}}\left\{\frac{e^{-\pi s}}{s^2+1}(s) + \frac{s}{s^2+1}\right\}$$

$$\Rightarrow \frac{1}{1-e^{-2\pi s}}\left\{\frac{s e^{-\pi s}+s}{s^2+1}\right\} \Rightarrow \frac{1}{1-e^{-2\pi s}}\left\{\frac{s(1+e^{-\pi s})}{s^2+1}\right\}$$

$$\Rightarrow \frac{1}{(1-e^{-\pi s})(1+e^{-\pi s})}\left\{\frac{s(1+e^{-\pi s})}{s^2+1}\right\}$$

$$\Rightarrow \frac{s}{(s^2+1)(1-e^{-\pi s})}$$

H.W

1. Find the Laplace transform of $f(t) = \begin{cases} \sin t & 0 < t < \pi \\ 0 & \pi < t < 2\pi \end{cases}$

Ans: $\dfrac{1}{(s^2+1)(1-e^{-\pi s})}$

2. Find the Laplace Transform of $f(t) = \begin{cases} t & 0 < t < \pi \\ 2\pi - t & \pi < t < 2\pi \end{cases}$

Refer Problem no. 4 & Replace $a = \pi$ & we get, $\frac{1}{s^2}\tanh\left(\frac{\pi s}{2}\right)$

3. Find the L.T of rectangular wave is given by

$$f(t) = \begin{cases} 1 & 0 < t < b \\ -1 & b < t < 2b \end{cases}$$

Replace $K = 1$ in Problem 1, we get $\frac{1}{s}\tanh\left(\frac{bs}{2}\right)$

4. Find the laplace transform of $f(t)$ is given by

$$f(t) = \begin{cases} 1 & 0 < t < a/2 \\ -1 & a/2 < t < a \end{cases}$$

Replace $E = 1$ in Problem 3, we get $\frac{1}{s}\tanh\left(\frac{as}{4}\right)$.

5. Find the Laplace Transform of $f(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & 1 < t < 2 \end{cases}$.

Ans: $L[f(t)] = \frac{1}{s(1+e^{-s})}$

LAPLACE TRANSFORMS OF DERIVATIVES.

* $L[f'(t)] = sL[f(t)] - f(0)$
* $L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0)$.
* $L[f'''(t)] = s^3 L[f(t)] - s^2 f(0) - sf'(0) - f''(0)$.

INITIAL VALUE THEOREM (I.V.T)

If $L[f(t)] = F(s)$, then $\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$.

Proof :-

W.K.T $L[f'(t)] = sL[f(t)] - f(0)$

$$L[f'(t)] = sF(s) - f(0)$$

$$\int_0^\infty e^{-st} f'(t)\,dt = sF(s) - f(0)$$

Apply $s \to \infty$ on both sides,

$$\lim_{s\to\infty} \int_0^\infty e^{-st} f'(t)\,dt = \lim_{s\to\infty} [sF(s) - f(0)] \qquad (e^{-\infty} = 0)$$

$$0 = \lim_{s\to\infty} sF(s) - f(0)$$

$$\therefore \lim_{s\to\infty} sF(s) = f(0) \qquad \Rightarrow \lim_{s\to\infty} sF(s) = \lim_{t\to 0} f(t).$$

FINAL VALUE THEOREM (F.V.T)

If $L[f(t)] = F(s)$, then $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

Proof:- W.K.T $L[f'(t)] = sF(s) - f(0)$

$$\int_0^\infty e^{-st} f'(t)\,dt = sF(s) - f(0)$$

Apply $s \to 0$ on both sides.

$$\lim_{s\to 0} [sF(s) - f(0)] = \lim_{s\to 0} \int_0^\infty e^{-st} f'(t)\,dt$$

$$\lim_{s\to 0} [sF(s) - f(0)] = \int_0^\infty f'(t)\,dt$$

$$= \int_0^\infty d[f(t)]$$

$$= [f(t)]_0^\infty$$

$$\lim_{s\to 0} sF(s) - f(0) = f(\infty) - f(0)$$

$$\therefore \lim_{s\to 0} sF(s) = f(\infty)$$

$$\therefore \lim_{s\to 0} sF(s) = \lim_{t\to\infty} f(t).$$

Problems based on I.V.T & F.V.T

①. Verify I.V.T & F.V.T $f(t) = 1+e^{-t}(\sin t+\cos t)$

Sol:-

I.V.T $\quad \underset{t\to 0}{lt}\, f(t) = \underset{s\to\infty}{lt}\, sF(s)$

L.H.S

$$\underset{t\to 0}{lt}\, f(t) = \underset{t\to 0}{lt}\left[1+e^{-t}(\sin t+\cos t)\right]$$

$$= 1+e^{0}(\sin 0+\cos 0)$$

$$= 1+(0+1)$$

$$= 2 \quad \text{— ①}$$

R.H.S

$$f(t) = 1+e^{-t}\sin t+e^{-t}\cos t$$

$$L[f(t)] = L[1]+L[e^{-t}\sin t]+L[e^{-t}\cos t]$$

$$F(s) = \frac{1}{s}+\left\{L[\sin t]\right\}_{s\to s+1}+\left\{L[\cos t]\right\}_{s\to s+1}$$

$$F(s) = \frac{1}{s}+\left\{\frac{1}{s^2+1}\right\}_{s\to s+1}+\left\{\frac{s}{s^2+1}\right\}_{s\to s+1}$$

$$F(s) = \frac{1}{s}+\frac{1}{(s+1)^2+1}+\frac{s+1}{(s+1)^2+1}$$

$$F(s) = \frac{1}{s}+\frac{s+2}{s^2+2s+2}$$

Multiply by 's'

$$sF(s) = \frac{s}{s}+\frac{s(s+2)}{s^2+2s+2} \Rightarrow 1+\frac{s^2+2s}{s^2+2s+2} \quad \text{— ⊛}$$

Apply $\underset{s\to\infty}{lt}$ on b/s:

$$\underset{s\to\infty}{lt}\, sF(s) = \underset{s\to\infty}{lt}\left[1+\frac{s^2(1+2s/s^2)}{s^2(1+2s/s^2+2/s^2)}\right]$$

$$= \underset{s\to\infty}{lt}\left[1+\frac{(1+2s/s^2)}{(1+2s/s^2+2/s^2)}\right]$$

$$= 1+\frac{(1+0)}{(1+0)} \Rightarrow 1+1 \Rightarrow 2 \quad \text{— ②}$$

from ① & ② are Equal. ∴ I.V.T Verified.

F.V.T

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

L.H.S

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty}\left[1+e^{-t}(\sin t+\cos t)\right] \Rightarrow \left[1+e^{-\infty}(\sin t+\cos t)\right]$$
$$=1+0$$
$$=1 \quad \text{—} (1)$$

R.H.S

$$(*) \Rightarrow \quad sF(s) = 1+\frac{s^2+2s}{s^2+2s+2}$$

Apply $s\to 0$ on b/s:

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[1+\frac{s^2+2s}{s^2+2s+2}\right] \Rightarrow \left[1+\frac{0}{2}\right] \Rightarrow 1 \quad \text{—} (2)$$

from (1) & (2) are Equal. L.H.S = R.H.S.

∴ F.V.T Verified.

2. Verify I.V.T & F.V.T for $f(t) = e^{-t}(t+2)^2$.

Sol:-

I.V.T : $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

L.H.S $\quad \lim_{t\to 0} f(t) = \lim_{t\to 0} e^{0}(0+2)^2 \Rightarrow 4 \quad \text{—} (1)$

R.H.S.

$$f(t) = e^{-t}(t+2)^2$$
$$f(t) = e^{-t}(t^2+4t+4)$$
$$L[f(t)] = L\left[e^{-t}t^2+4te^{-t}+4e^{-t}\right]$$
$$F(s) = L\left[t^2e^{-t}\right]+4L\left[te^{-t}\right]+4L\left[e^{-t}\right]$$
$$F(s) = \left\{L\left[t^2\right]\right\}_{s\to s+1}+4\left\{L[t]\right\}_{s\to s+1}+4L\left[e^{-t}\right]$$
$$F(s) = \left\{\frac{2}{s^3}\right\}_{s\to s+1}+4\left\{\frac{1}{s^2}\right\}_{s\to s+1}+4\left(\frac{1}{s+1}\right)$$
$$F(s) = \frac{2}{(s+1)^3}+\frac{4}{(s+1)^2}+\frac{4}{s+1}$$

X'ly by s:

$$sF(s) = \frac{2s}{(s+1)^3} + \frac{4s}{(s+1)^2} + \frac{4s}{s+1} \quad \text{—} (*)$$

Apply $\underset{s\to\infty}{lt}$ on b/s:

$$\underset{s\to\infty}{lt} sF(s) = \underset{s\to\infty}{lt}\left[\frac{2s}{s^3(1+1/s)^3} + \frac{4s}{s^2(1+1/s)^2} + \frac{4s}{s(1+1/s)}\right]$$

$$= \underset{s\to\infty}{lt}\left[\frac{2}{s^2(1+1/s)^3} + \frac{4}{s(1+1/s)^2} + \frac{4}{(1+1/s)}\right] \qquad (\because \tfrac{1}{\infty}=0)$$

$$\Rightarrow 0+0+4$$

$$\Rightarrow 4 \quad \text{—} (2)$$

from (1) & (2) are Equal. $\therefore$ I.V.T are Verified.

F.V.T $\underset{t\to\infty}{lt} f(t) = \underset{s\to 0}{lt} sF(s)$

$(\because e^{-\infty}=0)$

L.H.S $\underset{t\to\infty}{lt} f(t) = \underset{t\to\infty}{lt} e^{-t}(t+2)^2 \Rightarrow 0 \quad \text{—} (1)$

(or) $\underset{t\to\infty}{lt} \frac{(t+2)^2}{e^t} = \infty/\infty$

$\Rightarrow \underset{t\to\infty}{lt} \frac{2(t+2)}{e^t} \Rightarrow \infty/\infty$

$\Rightarrow \underset{t\to\infty}{lt} 2/e^t = 2/\infty = 0$

R.H.S

$$(*) \Rightarrow sF(s) = \frac{2s}{(s+1)^3} + \frac{4s}{(s+1)^2} + \frac{4s}{(s+1)}$$

Apply $s\to 0$ on b/s:

$$\underset{s\to 0}{lt} sF(s) = \underset{s\to 0}{lt}\left[\frac{2s}{(s+1)^3} + \frac{4s}{(s+1)^2} + \frac{4s}{(s+1)}\right]$$

$$\Rightarrow \frac{0}{1} + \frac{0}{1} + \frac{0}{1} \Rightarrow 0. \quad \text{—} (2)$$

from (1) & (2) are Equal. L.H.S = R.H.S

F.V.T are Verified.

H.W

1. Verify I.V.T $f(t) = ae^{-bt}$ Ans: a

2. Verify F.V.T $f(t) = 1 - e^{-at}$ Ans: 1

DERIVATIVES OF TRANSFORMS : (MULTIPLICATION OF "t")

If $L[f(t)] = F(s)$, then

$$L[t\,f(t)] = -\frac{d}{ds} L[f(t)]$$

$$L[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2} L[f(t)]$$

$\therefore$ In General,

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} L[f(t)] \quad \text{(or)} \quad (-1)^n \frac{d^n}{ds^n} F(s).$$

Problems:-

① Find $L[t\sin at]$ & $L[t\cos at]$

Sol:-

* $L[t\sin at] = -\frac{d}{ds} L[\sin at]$

$$= -\frac{d}{ds}\left[\frac{a}{s^2+a^2}\right]$$

$$= -\frac{d}{ds}\left[a(s^2+a^2)^{-1}\right]$$

$$= -\left[a(-1)(s^2+a^2)^{-2}(2s)\right]$$

$$= 2as\,(s^2+a^2)^{-2}$$

$$= \frac{2as}{(s^2+a^2)^2}$$

* $L[t\cos at] = -\frac{d}{ds} L[\cos at]$

$$= -\frac{d}{ds}\left[\frac{s}{s^2+a^2}\right]$$

$$= -\frac{d}{ds}\left[\frac{s}{s^2+a^2}\right] \qquad d\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

$$\Rightarrow -\left[\frac{(s^2+a^2)(1) - s(2s)}{(s^2+a^2)^2}\right]$$

$$\Rightarrow -\left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right] \Rightarrow \frac{s^2-a^2}{(s^2+a^2)^2}$$

② Find $L[te^{-3t}\cos 2t]$

Sol:-

$$L[t\cos at] = \frac{s^2-a^2}{(s^2+a^2)^2}$$

$$L[t\cos 2t] = \frac{s^2-2^2}{(s^2+4)^2} \Rightarrow \frac{s^2-4}{(s^2+4)^2}$$

$$\therefore L[te^{-3t}\cos 2t] = \left\{L[t\cos 2t]\right\}_{s\to s+3}$$

$$= \left\{\frac{s^2-4}{(s^2+4)^2}\right\}_{s\to s+3} \Rightarrow \frac{(s+3)^2-4}{\left[(s+3)^2+4\right]^2}$$

3. $L[t \sin 3t \cos 2t]$

Sol:- $\sin 3t \cos 2t = \frac{1}{2}[\sin 5t + \sin t]$

$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

$\therefore L[t\sin 3t \cos 2t] = -\frac{d}{ds} L[\sin 3t \cos 2t]$

$= -\frac{d}{ds} L\left[\frac{1}{2}(\sin 5t + \sin t)\right]$

$= -\frac{1}{2}\frac{d}{ds}[L(\sin 5t) + L(\sin t)]$

$= -\frac{1}{2}\left\{\frac{d}{ds}\left[\frac{5}{s^2+25} + \frac{1}{s^2+1}\right]\right\}$

$= -\frac{1}{2}\left\{\frac{d}{ds}\left[5(s^2+25)^{-1} + (s^2+1)^{-1}\right]\right\}$

$\Rightarrow -\frac{1}{2}\left\{5(-1)(s^2+25)^{-2}(2s) + (-1)(s^2+1)^{-2}(2s)\right\}$

$= -\frac{1}{2}\left\{\frac{-10s}{(s^2+25)^2} - \frac{2s}{(s^2+1)^2}\right\} \Rightarrow \frac{5s}{(s^2+25)^2} + \frac{s}{(s^2+1)^2}$ //.

4. Find (i) $L[t^2 \cos t]$ (ii) $L[t^2 e^{-t} \cos t]$.

Sol:- $L[t^2\cos t] = (-1)^2 \frac{d^2}{ds^2} L[\cos t] \Rightarrow 1 \frac{d^2}{ds^2}\left[\frac{s}{s^2+1}\right]$

$d\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$

$= \frac{d}{ds}\left\{\frac{(s^2+1)(1) - s(2s)}{(s^2+1)^2}\right\}$

$= \frac{d}{ds}\left\{\frac{1-s^2}{(s^2+1)^2}\right\}$

$= \frac{(s^2+1)^2(-2s) - (1-s^2)\,2(s^2+1)(2s)}{(s^2+1)^4}$

$= (s^2+1)\left\{\frac{(s^2+1)(-2s) - (1-s^2)(4s)}{(s^2+1)^4}\right\} \Rightarrow \frac{-2s^3 - 2s - 4s + 4s^3}{(s^2+1)^3}$

$L[t^2\cos t] \Rightarrow \frac{2s^3 - 6s}{(s^2+1)^3}$

$\therefore L[e^{-t} t^2 \cos t] = \{L[t^2\cos t]\}_{s \to s+1} = \left\{\frac{2s^3-6s}{(s^2+1)^3}\right\}_{s\to s+1} \Rightarrow \frac{2(s+1)^3 - 6(s+1)}{[(s+1)^2+1]^3}$ //.

5. Find the Laplace Transform of $[t\cos t \sinh 2t]$

Sol:-

$$L[t\cos t\sinh 2t] = L\left[t\cos t\left(\frac{e^{2t}-e^{-2t}}{2}\right)\right]$$

$$\sinh t = \frac{e^{t}-e^{-t}}{2}$$

$$\cosh t = \frac{e^{t}+e^{-t}}{2}$$

$$= \frac{1}{2}\left\{L\left[t\cos t\,(e^{2t}-e^{-2t})\right]\right\}$$

$$= \frac{1}{2}\left\{L\left[t\cos t\, e^{2t}\right] - L\left[t\cos t\, e^{-2t}\right]\right\} \quad \text{—①}$$

$$= \frac{1}{2}L\left[e^{2t}t\cos t\right] - \frac{1}{2}L\left[t\cos t\, e^{-2t}\right]$$

$$\therefore L\left[e^{2t}t\cos t\right] = \left\{L[t\cos t]\right\}_{s\to s-2} \qquad \left(\because L[t\cos at] = \frac{s^2-a^2}{(s^2+a^2)^2}\right)$$

$$= \left[\frac{s^2-1}{(s^2+1)^2}\right]_{s\to s-2} \Rightarrow \frac{(s-2)^2-1}{[(s-2)^2+1]^2} \Rightarrow \frac{s^2-4s+4-1}{[s^2-4s+4+1]^2}$$

$$L\left[e^{2t}t\cos t\right] \Rightarrow \frac{s^2-4s+3}{(s^2-4s+5)^2}$$

$$L\left[e^{-2t}t\cos t\right] = \left\{L[t\cos t]\right\}_{s\to s+2} \Rightarrow \left\{\frac{s^2-1}{(s^2+1)^2}\right\}_{s\to s+2}$$

$$\Rightarrow \frac{(s+2)^2-1}{[(s+2)^2+1]^2} \Rightarrow \frac{s^2+4s+3}{(s^2+4s+5)^2}.$$

$$\text{①} \Rightarrow L[t\cos t\sinh 2t] = \frac{1}{2}\left[\frac{s^2-4s+3}{(s^2-4s+5)^2}\right] - \frac{1}{2}\left[\frac{s^2+4s+3}{(s^2+4s+5)^2}\right] //$$

INTEGRALS OF TRANSFORMS: (Division by "t")

If $L[f(t)] = F(s)$, & $\lim_{t\to 0}\frac{f(t)}{t}$ = finite limit, then

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty L[f(t)]\,ds.$$

$$L\left[\frac{f(t)}{t^2}\right] = \int_s^\infty\int_s^\infty L[f(t)]\,ds\,ds$$

& so on.

Problems:-

① Find $L\left[\frac{e^{at}-e^{-bt}}{t}\right]$

Sol:- $\lim_{t\to 0} \frac{e^{at}-e^{-bt}}{t} = \frac{e^{0}-e^{0}}{0} \Rightarrow \frac{1-1}{0} \Rightarrow 0/0$ (undefined form)

Apply L'Hopital Rule

$\lim_{t\to 0} \frac{ae^{at}+be^{-bt}}{1} = ae^{0}+be^{0} \Rightarrow (a+b)$ (finite limit

$$\therefore L\left[\frac{e^{at}-e^{-bt}}{t}\right] = \int_{s}^{\infty} L\left[e^{at}-e^{-bt}\right] ds$$

$$= \int_{s}^{\infty} \left[\frac{1}{s-a} - \frac{1}{s+b}\right] ds \Rightarrow \left[\log(s-a)-\log(s+b)\right]_{s}^{\infty}$$

$\log A - \log B = \log(A/B)$

$$= \left[\log\left(\frac{s-a}{s+b}\right)\right]_{s}^{\infty}$$

$$= \left[\log \frac{\cancel{s}(1-a/s)}{\cancel{s}(1+b/s)}\right]_{s}^{\infty} \Rightarrow \left[\log\left(\frac{1-a/s}{1+b/s}\right)\right]_{s}^{\infty}$$

$$\Rightarrow \log 1 - \log\left(\frac{1-a/s}{1+b/s}\right) \Rightarrow 0 - \log\left(\frac{\frac{s-a}{s}}{\frac{s+b}{s}}\right)$$

$$\Rightarrow -\log\left(\frac{s-a}{s+b}\right) \Rightarrow \log\left(\frac{s-a}{s+b}\right)^{-1}$$

$\log a^{m} = m\log a$.

$$\Rightarrow \log\left(\frac{s+b}{s-a}\right)_{\!/\!/}$$

2. Find $L\left[\frac{\cos at}{t}\right]$ (or) Does $L\left[\frac{\cos at}{t}\right]$ exists.

Sol:- $\lim_{t\to 0} \frac{f(t)}{t} = \lim_{t\to 0} \frac{\cos at}{t} \Rightarrow \frac{\cos 0}{0} \Rightarrow 1/0 \Rightarrow \infty$ (Not finite)

$\therefore \lim_{t\to 0} \frac{f(t)}{t}$ does not exists

$\therefore L\left[\frac{\cos at}{t}\right]$ does not exists.

③ Find $L\left[\frac{\cos at - \cos bt}{t}\right]$.

Sol:: $\lim_{t\to 0} \frac{\cos at - \cos bt}{t} = \frac{1-1}{0} \Rightarrow 0/0$ (Indetermined form)

Apply L'Hopital Rule

$\lim_{t\to 0}\left[\frac{a(-\sin at) - b(-\sin bt)}{1}\right] = 0$ (finite value)

$$\therefore\ L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^\infty L[\cos at - \cos bt]\,ds$$

$$= \int_s^\infty \left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right] ds$$

$$= \frac{1}{2}\times 2\int_s^\infty \left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right] ds$$

$$= \frac{1}{2}\left\{\int_s^\infty \left[\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right] ds\right\}$$

$$= \frac{1}{2}\left\{\log(s^2+a^2) - \log(s^2+b^2)\right\}_s^\infty$$

$$= \frac{1}{2}\left\{\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\}_s^\infty$$

$$= \frac{1}{2}\left\{\log\left(\frac{s^2+b^2}{s^2+a^2}\right)\right\}.$$

$\int \frac{f'(x)}{f(x)}dx = \log f(x)$

$\log 1 = 0$

$\log A - \log B = \log\left(\frac{A}{B}\right)$

④ Find $L\left[\frac{1-\cos at}{t}\right]$

Sol:: $\lim_{t\to 0} \frac{1-\cos at}{t} = \frac{1-1}{0} \Rightarrow 0/0$

Apply L'Hopital Rule $\lim_{t\to 0} \frac{a\sin at}{1} = 0$ (finite value)

$$\therefore L\left[\frac{1-\cos at}{t}\right] = \int_s^\infty L[1-\cos at]\,ds \Rightarrow \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+a^2}\right] ds$$

$$\Rightarrow \left[\log s - \frac{1}{2}\log(s^2+a^2)\right]_s^\infty$$

$\log f(x) = \int \frac{f'(x)}{f(x)}dx$

$$\Rightarrow \left[\log s - \log\sqrt{s^2+a^2}\right]_s^\infty \Rightarrow \left[\log \frac{s}{\sqrt{s^2+a^2}}\right]_s^\infty$$

$$\Rightarrow \log\left[\frac{\sqrt{s^2+a^2}}{s}\right].$$

⑤ Find $L\left[\frac{\sin at}{t}\right]$

Sol:- $\underset{t\to 0}{lt} \frac{\sin at}{t} = \frac{\sin 0}{0} \Rightarrow 0/0$ (form)

Apply L'Hopital Rule

$\underset{t\to 0}{lt} \frac{a\cos at}{1} = a$ (finite value)

$\therefore L\left[\frac{\sin at}{t}\right] = \int_s^\infty L[\sin at]\,ds \Rightarrow \int_s^\infty \frac{a}{s^2+a^2}\,ds$

$\Rightarrow a\left[\frac{1}{a}\tan^{-1}(s/a)\right]_s^\infty$

$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

$= \left[\tan^{-1}(s/a)\right]_s^\infty$

$= \tan^{-1}(\infty) - \tan^{-1}(s/a)$

$= \pi/2 - \tan^{-1}(s/a)$

$= \tan^{-1}(a/s)$

Note:-

* $\tan^{-1}\left(\frac{s}{a}\right) + \cot^{-1}(s/a) = \pi/2$. & * $\cot^{-1}\left(\frac{s}{a}\right) = \tan^{-1}\left(\frac{a}{s}\right)$

⑥ Find $L\left[\frac{1-\cos t}{t^2}\right]$

Sol:-

$L\left[\frac{1-\cos t}{t^2}\right] = \int_s^\infty \left\{\int_s^\infty L[1-\cos t]\,ds\right\} ds$ —— (*)

Consider $\int_s^\infty L[1-\cos t]\,ds = \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+1}\right] ds$

$= \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$

$= \left[\log s - \log\sqrt{s^2+1}\right]_s^\infty$

$= \left[\log \frac{s}{\sqrt{s^2+1}}\right]_s^\infty \Rightarrow \left(\log \frac{\sqrt{s^2+1}}{s}\right)$

$= \left[\log \frac{(s^2+1)^{1/2}}{(s^2)^{1/2}}\right] \Rightarrow \left\{\log\left[\frac{s^2+1}{s^2}\right]^{1/2}\right\}$

$\int_s^\infty L[1-\cos t]\,ds = \frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right)$

(*) $\Rightarrow L\left[\frac{1-\cos t}{t^2}\right] = \int_s^\infty \frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right)ds$

$$= \frac{1}{2}\int_s^\infty \underbrace{\log(1+1/s^2)}_{u}\,\underbrace{ds}_{dv}$$

Integration by Parts

$\int u\,dv = uv - \int v\,du$

$u = \log(1+1/s^2)$ $\quad dv = ds$

$du = \frac{1}{1+1/s^2}\left(-\frac{2}{s^3}\right)$ $\quad v = s$

$$= \frac{1}{2}\left\{\left[\left(\log(1+1/s^2)\,s\right)\right]_s^\infty - \int_s^\infty s\cdot\frac{1}{1+1/s^2}\left(-\frac{2}{s^3}\right)ds\right\}$$

$$= \frac{1}{2}\left\{\left[0 - s\log(1+1/s^2)\right] + 2\int_s^\infty \frac{s^2\cdot s}{s^2+1}\left(1/s^3\right)ds\right.$$

$$= \frac{1}{2}\left\{-s\log(1+1/s^2) + 2\int_s^\infty \frac{ds}{s^2+1}\right\}$$

$$= -\frac{1}{2}s\log\left(\frac{s^2+1}{s^2}\right) + \frac{2}{2}\left\{\tan^{-1}(s)\right\}_s^\infty$$

$\tan^{-1}(\infty) = \pi/2$

$$\Rightarrow -\frac{1}{2}s\log\left(\frac{s^2+1}{s^2}\right) + \left\{\pi/2 - \tan^{-1}(s)\right\}$$

$(\because \tan^{-1}(s) + \cot^{-1}s = \pi/2$.

$$\Rightarrow -\frac{1}{2}s\log\left(\frac{s^2+1}{s^2}\right) + \cot^{-1}(s),$$

H.W

1. Find $L\left[\frac{e^{-t}\sin t}{t}\right]$ Ans: $\cot^{-1}(s+1)$

2. Find $L\left[\frac{\sin^2 t}{t}\right]$ Ans: $\frac{1}{2}\log\left(\sqrt{\frac{s^2+4}{s}}\right)$

   Hints: $\sin^2 t = \frac{1-\cos 2t}{2}$

3. Find $L\left[\frac{\cos 2t - \cos 3t}{t}\right]$ Ans: $\frac{1}{2}\log\left(\frac{s^2+4}{s^2+9}\right)$

4. Find $L\left[\frac{\sin at}{t}\right]$. Hence, show that $\int_0^\infty \frac{\sin t}{t}dt = \pi/2$.

Evaluation of improper integrals Using Laplace Transform

TYPE 1:- Integrals of the form $\int_0^\infty$

① Evaluate $\int_0^\infty t\,e^{-2t}\cos t\,dt$

Sol:- $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

We have $\int_0^\infty t\,e^{-2t}\cos t\,dt = \left\{L[t\cos t]\right\}_{s=2}$

$$= \left\{-\frac{d}{ds}L[\cos t]\right\}_{s=2} \Rightarrow \left\{-\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right\}_{s=2}$$

$$\Rightarrow -\left[\frac{(s^2+1)1 - s\cdot 2s}{(s^2+1)^2}\right]_{s=2} \Rightarrow \left[\frac{s^2-1}{(s^2+1)^2}\right]_{s=2}$$

$$\Rightarrow \frac{4-1}{(5)^2} \Rightarrow 3/25 .$$

② Evaluate $\int_0^\infty \left(\frac{\cos at - \cos bt}{t}\right) dt.$

Sol:- $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$$\int_0^\infty e^{0t}\left(\frac{\cos at - \cos bt}{t}\right)dt = \left\{L\left[\frac{\cos at - \cos bt}{t}\right]\right\}_{s=0}$$

$$= \left\{\int_s^\infty L[\cos at - \cos bt]\,ds\right\}_{s=0}$$

$$= \left\{\int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right)ds\right\}_{s=0}$$

$$\Rightarrow \left\{\left[\frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right]_s^\infty\right\}_{s=0}$$

$$\Rightarrow \left\{\left[\frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^\infty\right\}_{s=0}$$

$$\Rightarrow \left\{\frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)\right\}_{s=0} \Rightarrow \frac{1}{2}\log\left(\frac{b^2}{a^2}\right) \Rightarrow \frac{1}{2}\log(b/a)^2$$

$$\Rightarrow \frac{1}{2}\cdot 2\log(b/a)$$

$$\Rightarrow \log(b/a)$$

3. Evaluate $\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}\,dt$.

Sol:- W.K.T $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$$\int_0^\infty e^{-0t}\left(\frac{e^{-t}-e^{-3t}}{t}\right)dt = \left\{L\left[\frac{e^{-t}-e^{-3t}}{t}\right]\right\}_{s=0}$$

$$= \left\{\int_s^\infty L[e^{-t}-e^{-3t}]\,ds\right\}_{s=0}$$

$$= \left\{\int_s^\infty \left[\frac{1}{s+1} - \frac{1}{s+3}\right]ds\right\}_{s=0} \Rightarrow \left\{\left[\log(s+1) - \log(s+3)\right]_s^\infty\right\}_{s=0}$$

$$\Rightarrow \left\{\left[\log\left(\frac{s+1}{s+3}\right)\right]_s^\infty\right\}_{s=0}$$

$$\Rightarrow \left\{\log\left(\frac{s+3}{s+1}\right)\right\}_{s=0} \Rightarrow \log\left(\frac{0+3}{0+1}\right) \Rightarrow \log 3.$$

4. Evaluate $\int_0^\infty e^{-t}\left(\frac{1-\cos t}{t}\right)dt$

Sol:- $$\int_0^\infty e^{-t}\left(\frac{1-\cos t}{t}\right)dt = \left\{L\left[\frac{1-\cos t}{t}\right]\right\}_{s=1} \quad \text{—} (*)$$

Consider $L\left[\frac{1-\cos t}{t}\right] = \int_s^\infty L[1-\cos t]\,ds \Rightarrow \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+1}\right]ds$

$$= \left\{\log s - \tfrac{1}{2}\log(s^2+1)\right\}_s^\infty \qquad \log a^m = m\log a$$

$$= \left\{\log(s^2)^{1/2} - \tfrac{1}{2}\log(s^2+1)\right\}_s^\infty$$

$$= \left\{\tfrac{1}{2}\log s^2 - \tfrac{1}{2}\log(s^2+1)\right\}_s^\infty$$

$$= \left\{\tfrac{1}{2}\log\left(\frac{s^2}{s^2+1}\right)\right\}_s^\infty \Rightarrow \tfrac{1}{2}\log\left(\frac{s^2+1}{s^2}\right)$$

$$(*) \Rightarrow \left\{L\left[\frac{1-\cos t}{t}\right]\right\}_{s=1} = \left\{\tfrac{1}{2}\log\left(\frac{s^2+1}{s^2}\right)\right\}_{s=1}$$

$$= \tfrac{1}{2}\log\left(\frac{1+1}{1}\right)$$

$$= \tfrac{1}{2}\log 2 \;\; (\text{or}) \;\; \log\sqrt{2}$$

5. Evaluate $\int_0^\infty \frac{\sin at}{t}\,dt$

Sol:- $\int_0^\infty e^{-0t}\left(\frac{\sin at}{t}\right)dt = \left\{L\left[\frac{\sin at}{t}\right]\right\}_{s=0}$ —

↓ Refer P.No:33

$$= \left\{\pi/2 - \tan^{-1}(s/a)\right\}_{s=0} \Rightarrow \left\{\frac{\pi}{2} - 0\right\} \Rightarrow \pi/2.$$

6. Evaluate $\int_0^\infty e^{-t}\,\frac{\sin^2 t}{t}\,dt.$

Sol:- $\int_0^\infty e^{-t}\,\frac{\sin^2 t}{t}\,dt = \left\{L\left[\frac{\sin^2 t}{t}\right]\right\}_{s=1}$ — (*)

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

Now, Consider

$$L\left[\frac{\sin^2 t}{t}\right] = L\left[\frac{1-\cos 2t}{2t}\right]$$

$$= \frac{1}{2}L\left[\frac{1-\cos 2t}{t}\right]$$

$$= \frac{1}{2}\left\{\int_s^\infty L[1-\cos 2t]\,ds\right.$$

$$= \frac{1}{2}\left\{\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right)ds\right\} \Rightarrow \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$\Rightarrow \frac{1}{2}\left[\log s - \log\sqrt{s^2+4}\right]_s^\infty$$

$$\Rightarrow \left[\frac{1}{2}\log\left(\frac{s}{\sqrt{s^2+4}}\right)\right]_s^\infty$$

$$L\left[\frac{\sin^2 t}{t}\right] \Rightarrow \frac{1}{2}\log\frac{\sqrt{s^2+4}}{s}$$

$$(*) \Rightarrow \left\{L\left[\frac{\sin^2 t}{t}\right]\right\}_{s=1} = \left\{\frac{1}{2}\log\frac{\sqrt{s^2+4}}{s}\right\}_{s=1}$$

$$= \frac{1}{2}\log\sqrt{5}.$$

H.W:

1. Evaluate $\int_0^\infty e^{-t}\left(\frac{\cos 2t - \cos 3t}{t}\right)dt$ Ans: $\frac{1}{2}\log_e 2.$

2. Evaluate $\int_0^\infty t\,e^{3t}\cos 2t\,dt$ Ans: $\frac{5}{169}$

TYPE II :- Integrals of the type $\int_0^t$

① Evaluate $\int_0^t t e^{-t} \sin t \, dt$

Sol:- $L\left[\int_0^t t e^{-t} \sin t \, dt\right] = \frac{1}{s} L\left[t e^{-t} \sin t\right]$

$= \frac{1}{s}\left\{L[t \sin t]\right\}_{s \to s+1}$

$= \frac{1}{s}\left\{\frac{2s}{(s^2+1)^2}\right\}_{s \to s+1} \Rightarrow \frac{1}{s}\left\{\frac{2(s+1)}{[(s+1)^2+1]^2}\right\}$

$\Rightarrow \frac{1}{s}\left\{\frac{2(s+1)}{(s^2+2s+2)^2}\right\}.$

② Evaluate $e^{-4t}\int_0^t t \sin 3t \, dt.$

Sol:- $L\left[e^{-4t}\int_0^t t \sin 3t \, dt\right] = \left\{L\left[\int_0^t t \sin 3t \, dt\right]\right\}_{s \to s+4}$

$= \left\{\frac{1}{s} L[t \sin 3t]\right\}_{s \to s+4}$

$= \left\{\frac{1}{\cancel{s}}\left(\frac{6\cancel{s}}{(s^2+9)^2}\right)\right\}_{s \to s+4}$

$\Rightarrow \left\{\frac{6}{[(s+4)^2+9]^2}\right\} \Rightarrow \left\{\frac{6}{(s^2+8s+25)^2}\right\}.$

$L[t \sin at] = \frac{2as}{(s^2+a^2)}$

$a=3$

③ Evaluate $\int_0^t \frac{e^{-t}\sin t}{t} \, dt.$

Sol:- $L\left[\int_0^t \frac{e^{-t}\sin t}{t} dt\right] = \frac{1}{s}\left[L\left(\frac{e^{-t}\sin t}{t}\right)\right]$

$= \frac{1}{s}\left\{\left[L\left(\frac{\sin t}{t}\right)\right]_{s \to s+1}\right\}$

↓ Refer P.No: 33

$= \frac{1}{s}\left\{\left[\cot^{-1}(s)\right]_{s \to s+1}\right\}$

$= \frac{1}{s}\left\{\cot^{-1}(s+1)\right\}.$

$\left(\because L\left[\frac{\sin at}{t}\right]\right.$

$= \frac{\pi}{2} - \tan^{-1}(s/a)$

(or)

$= \tan^{-1}(a/s)$

(or)

$= \cot^{-1}(s/a)$

H.W

4. Evaluate $e^{-t}\int_0^t t \cos t \, dt$

Ans: $\frac{s^2+2s}{(s+1)(s^2+2s+2)^2}.$

INVERSE LAPLACE TRANSFORM

If $L[f(t)] = F(s)$, then $L^{-1}[F(s)] = f(t)$.

For Ex:

$$L[e^{at}] = \frac{1}{s-a} \quad \text{then} \quad L^{-1}\left[\frac{1}{s-a}\right] = e^{at}.$$

Important Formula:-

1. $L^{-1}\left[\frac{1}{s}\right] = 1$
2. $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$
3. $L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$
4. $L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$
5. $L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!}$
6. $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a}$
7. $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$
8. $L^{-1}\left[\frac{1}{(s-a)^2}\right] = e^{at}\,t$
9. $L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{\sinh at}{a}$
10. $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$
11. $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t\sin at}{2a}$
12. $L^{-1}\left[\frac{s^2-a^2}{(s^2+a^2)^2}\right] = t\cos at$
13. $L^{-1}\left[\frac{s}{(s^2-a^2)^2}\right] = \frac{t\sinh at}{2a}$
14. $L^{-1}\left[\frac{s^2+a^2}{(s^2-a^2)^2}\right] = t\cosh at$
15. $L^{-1}[1] = \delta(t)$.

Problems:-

① $L^{-1}\left[\frac{1}{s^2} + \frac{1}{s+4} + \frac{1}{s^2+4} + \frac{s}{s^2-9} + \frac{1}{s^2-25}\right]$

Sol:- $L^{-1}\left[\frac{1}{s^2}\right] + L^{-1}\left[\frac{1}{s+4}\right] + L^{-1}\left[\frac{1}{s^2+4}\right] + L^{-1}\left[\frac{s}{s^2-9}\right] + L^{-1}\left[\frac{1}{s^2-25}\right]$

$$\Rightarrow \quad t + e^{-4t} + \frac{\sin 2t}{2} + \cosh 3t + \frac{\sinh 5t}{5}$$

2. $L^{-1}\left[\frac{1}{3s-7}\right]$

Sol:- $L^{-1}\left[\frac{1}{3s-7}\right] = L^{-1}\left[\frac{1}{3(s-7/3)}\right]$

$= \frac{1}{3}L^{-1}\left[\frac{1}{s-7/3}\right] \Rightarrow \frac{1}{3}e^{7/3\,t}$

3. $L^{-1}\left[\frac{1}{(s-2)^2+1}\right]$

Sol:- $L^{-1}\left[\frac{1}{(s-2)^2+1}\right] = e^{2t}L^{-1}\left[\frac{1}{s^2+1}\right]$

$= e^{2t}\sin t.$

4. $L^{-1}\left[\frac{s+2}{s^2+4s+8}\right]$

Sol:- $L^{-1}\left[\frac{s+2}{s^2+4s+8}\right] = L^{-1}\left[\frac{s+2}{(s+2)^2+4}\right] \Rightarrow e^{-2t}L^{-1}\left[\frac{s}{s^2+4}\right]$

$\Rightarrow e^{-2t}\cos 2t$

5. $L^{-1}\left[\frac{s^2-3s+2}{s^3}\right]$

Sol:- $L^{-1}\left[\frac{s^2-3s+2}{s^3}\right] = L^{-1}\left[\frac{s^2}{s^3} - \frac{3s}{s^3} + \frac{2}{s^3}\right]$

$\Rightarrow L^{-1}\left[\frac{1}{s} - \frac{3}{s^2} + \frac{2}{s^3}\right] \Rightarrow L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{3}{s^2}\right] + 2L^{-1}\left[\frac{1}{s^3}\right]$

$\Rightarrow 1-3t+t^2.$

6. $L^{-1}\left[\frac{s}{(s+2)^2}\right]$

Sol:- $L^{-1}\left[\frac{s}{(s+2)^2}\right] = L^{-1}\left[\frac{s+2-2}{(s+2)^2}\right]$

$= L^{-1}\left[\frac{s+2}{(s+2)^2} - \frac{2}{(s+2)^2}\right] \Rightarrow L^{-1}\left[\frac{1}{s+2}\right] - 2L^{-1}\left[\frac{1}{(s+2)^2}\right]$

$= e^{-2t} - 2e^{-2t}L^{-1}\left[\frac{1}{s^2}\right]$

$= e^{-2t} - 2e^{-2t}t.$

7. $L^{-1}\left[\frac{s-3}{s^2+4s+13}\right]$

Sol:- $L^{-1}\left[\frac{s-3}{s^2+4s+13}\right] = L^{-1}\left[\frac{s-3}{(s+2)^2+13-4}\right]$

$$= L^{-1}\left[\frac{s-3}{(s+2)^2+9}\right]$$

$$= L^{-1}\left[\frac{s+2-5}{(s+2)^2+9}\right]$$

$$= L^{-1}\left[\frac{s+2}{(s+2)^2+9}\right] - L^{-1}\left[\frac{5}{(s+2)^2+9}\right]$$

$$= L^{-1}\left[\frac{s+2}{(s+2)^2+9}\right] - 5L^{-1}\left[\frac{1}{(s+2)^2+9}\right]$$

$$= e^{-2t}L^{-1}\left[\frac{s}{s^2+3^2}\right] - \frac{5}{3}L^{-1}\left[\frac{3}{(s+2)^2+9}\right]$$

$$= e^{-2t}\cos 3t - \frac{5}{3}e^{-2t}L^{-1}\left[\frac{3}{s^2+3^2}\right]$$

$$= e^{-2t}\cos 3t - \frac{5}{3}e^{-2t}\sin 3t.$$

Alternate:-

$L^{-1}\left[\frac{s}{(s+2)^2+9}\right] - L^{-1}\left[\frac{3}{(s+2)^2+9}\right]$

$\frac{d}{dt}L^{-1}\left[\frac{1}{(s+2)^2+9}\right] - e^{-2t}L^{-1}\left[\frac{3}{s^2+3^2}\right]$

$\Rightarrow \frac{d}{dt}\left[e^{-2t}\frac{\sin 3t}{3}\right] - e^{-2t}\sin 3t$

$\hookrightarrow d(uv)$

$\Rightarrow e^{-2t}\cos 3t - \frac{5}{3}e^{-2t}\sin 3t$

OMITTING by 's' (Multiplication of 's')

$L[f'(t)] = sL[f(t)] - f(0)$, Provided $f(0)=0$

$L^{-1}[sF(s)] = f'(t)$

$$= \frac{d}{dt}f(t) \Rightarrow \frac{d}{dt}L^{-1}[F(s)]$$

$\therefore L^{-1}[sF(s)] = \frac{d}{dt}L^{-1}[F(s)]$

$|||^{ly}$ $L^{-1}[s^2F(s)] = \frac{d^2}{dt^2}L^{-1}[F(s)]$

Problems:-

① $L^{-1}\left[\dfrac{s^2}{(s-1)^4}\right]$

Sol:-

$$L^{-1}\left[\frac{s^2}{(s-1)^4}\right] = \frac{d^2}{dt^2} L^{-1}\left[\frac{1}{(s-1)^4}\right]$$ (By first shifting theorem)

$$= \frac{d^2}{dt^2}\left\{e^t L^{-1}\left[\frac{1}{s^4}\right]\right\}$$

$$= \frac{d^2}{dt^2}\left\{e^t \frac{t^3}{3!}\right\}$$

$\downarrow$ $d(uv) = uv' + vu'$

$$= \frac{1}{6}\left[\frac{d}{dt}\left\{e^t (3t^2) + t^3 e^t\right\}\right]$$

$$= \frac{1}{6}\left\{\frac{d}{dt}(3t^2e^t) + \frac{d}{dt}(t^3e^t)\right\}$$

$$= \frac{1}{6}\left\{(3t^2 e^t + 6t e^t) + (t^3 e^t + 3t^2 e^t)\right\}$$

$$= \frac{1}{6}\left\{6t^2e^t + 6te^t + t^3e^t\right\}$$

$$= t^2e^t + te^t + \frac{t^3e^t}{6}.$$

②. $L^{-1}\left[\dfrac{s}{(s+2)^2+4}\right]$

Sol:

$$L^{-1}\left[\frac{s}{(s+2)^2+4}\right] = \frac{d}{dt}\left\{L^{-1}\left[\frac{1}{(s+2)^2+4}\right]\right\}$$

$$= \frac{d}{dt}\left\{e^{-2t} L^{-1}\left[\frac{1}{s^2+2^2}\right]\right\} \Rightarrow \frac{d}{dt}\left\{\frac{e^{-2t}}{2} L^{-1}\left(\frac{2}{s^2+2^2}\right)\right\}$$

$$= \frac{d}{dt}\left\{\frac{e^{-2t}}{2}\sin 2t\right\}$$

$$= \frac{1}{2}\frac{d}{dt}\left\{e^{-2t}\sin 2t\right\}$$

$\downarrow$ $d(uv) = uv' + vu'$.

$$= \frac{1}{2}\left\{e^{-2t}\, 2\cos 2t + \sin 2t\,(-2e^{-2t})\right\}$$

$$= e^{-2t}\cos 2t - e^{-2t}\sin 2t.$$

OMITTING by $\frac{1}{s}$ (DIVISION OF S)

$L[f(t)] = F(s)$, then $L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}F(s)$

ie., $L^{-1}\left[\frac{1}{s}F(s)\right] = \int_0^t L^{-1}[F(s)]\,dt = \int_0^t f(t)\,dt$

IIIly $L^{-1}\left[\frac{1}{s^2}F(s)\right] = \int_0^t\int_0^t L^{-1}[F(s)]\,dt\,dt.$

Problems:

1) $L^{-1}\left[\frac{1}{s(s+2)^3}\right]$

Sol:- $L^{-1}\left[\frac{1}{s}\cdot\frac{1}{(s+2)^3}\right] = \int_0^t L^{-1}\left[\frac{1}{(s+2)^3}\right]dt$

$= \int_0^t e^{-2t} L^{-1}\left[\frac{1}{s^3}\right]dt \Rightarrow \int_0^t e^{-2t}\frac{t^2}{2!}\,dt$

$= \frac{1}{2}\int_0^t t^2 e^{-2t}\,dt$

Apply Bernoulli's $\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \ldots\ldots$

$= \frac{1}{2}\left[t^2\frac{e^{-2t}}{-2} - 2t\left(\frac{e^{-2t}}{4}\right) + 2\left(\frac{e^{-2t}}{-8}\right)\right]_0^t$

$= \frac{1}{2}\left[-\frac{t^2e^{-2t}}{2} - \frac{te^{-2t}}{2} + \frac{e^{-2t}}{4}\right]_0^t$

$= \frac{1}{2}\left\{-\frac{t^2e^{-2t}}{2} - \frac{te^{-2t}}{2} - \frac{e^{-2t}}{4} - (0-0-1/4)\right]$

$= \frac{1}{2}\left\{-\frac{t^2e^{-2t}}{2} - \frac{te^{-2t}}{2} - e^{-2t}/4 + 1/4\right\}$

$\Rightarrow \frac{1}{8}\left\{-2t^2e^{-2t} - 2te^{-2t} - e^{-2t} + 1\right\}$

2. Find $L^{-1}\left[\frac{s^2+3}{s(s^2+9)}\right]$

Sol:- $L^{-1}\left[\frac{s^2+3}{s(s^2+9)}\right] = L^{-1}\left[\frac{s^2+9-6}{s(s^2+9)}\right]$

$$= L^{-1}\left[\frac{s^2+9}{s(s^2+9)} - \frac{6}{s(s^2+9)}\right]$$

$$\Rightarrow L^{-1}\left(\frac{1}{s} - \frac{6}{s(s^2+9)}\right) \Rightarrow L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left[\frac{6}{s(s^2+9)}\right]$$

$$\Rightarrow 1 - \int_0^t L^{-1}\left[\frac{6}{s^2+9}\right] dt$$

$$\Rightarrow 1 - 2\int_0^t L^{-1}\left[\frac{3}{s^2+3^2}\right] dt$$

$$\Rightarrow 1 - 2\int_0^t \sin 3t\, dt \Rightarrow 1 - 2\left(-\frac{\cos 3t}{3}\right)_0^t$$

$$\Rightarrow 1 + \frac{2}{3}\left(\cos 3t\right)_0^t$$

$$\Rightarrow 1 + \frac{2}{3}\left(\cos 3t - \cos 0\right) \Rightarrow 1 + \frac{2}{3}(\cos 3t - 1).$$

2. $L^{-1}\left[\frac{1}{s(s^2-2s+5)}\right]$

Sol:- $L^{-1}\left[\frac{1}{s(s^2-2s+5)}\right] = \int_0^t L^{-1}\left[\frac{1}{s^2-2s+5}\right] dt$

$$\Rightarrow \int_0^t L^{-1}\left[\frac{1}{(s-1)^2+4}\right] dt \Rightarrow \int_0^t \frac{1}{2} L^{-1}\left[\frac{2}{(s-1)^2+2^2}\right] dt$$

$$\Rightarrow \frac{1}{2}\int_0^t e^t L^{-1}\left[\frac{2}{s^2+2^2}\right] dt$$

$$\Rightarrow \frac{1}{2}\int_0^t e^t \sin 2t\, dt$$

$$\left(\because \int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]\right)$$

$a=1 \quad b=2$

$$\Rightarrow \frac{1}{2}\left[\frac{e^t}{1^2+2^2}(\sin 2t - 2\cos 2t)\right]_0^t$$

$$\Rightarrow \frac{1}{2}\left\{\frac{e^t}{5}(\sin 2t - 2\cos 2t) - \frac{e^0}{5}(\sin 0 - 2\cos 0)\right\}$$

$$\Rightarrow \frac{1}{2}\left\{\frac{e^t}{5}(\sin 2t - 2\cos 2t) + 2/5\right\}$$

$$\Rightarrow \frac{1}{10}\left\{e^t(\sin 2t - 2\cos 2t) + 2\right\}$$

INVERSE LAPLACE TRANSFORMS OF DERIVATIVES OF F(S) & LOGARITHMIC FUNCTIONS (SPECIAL FUNCTIONS).

$L[f(t)] = F(s)$, then $L[tf(t)] = -F'(s)$

ie., If $L^{-1}[F(s)] = f(t)$, then $L^{-1}[F'(s)] = -tf(t)$

$$\boxed{L^{-1}[F'(s)] = -t\,L^{-1}[F(s)]}$$

Problems:

① $L^{-1}\left[\dfrac{s+3}{(s^2+6s+13)^2}\right]$

Sol:- $L^{-1}[F'(s)] = -t\,L^{-1}[F(s)]$

Let $F'(s) = \dfrac{s+3}{(s^2+6s+13)^2}$

Integrating w.r.to 's'

$$\int F'(s)\,ds = \int \frac{(s+3)}{(s^2+6s+13)^2}\,ds$$

$$F(s) = \int \frac{(s+3)}{(s^2+6s+13)^2}\,ds \quad \text{——} ①$$

Put $t = s^2+6s+13$

$dt = (2s+6)\,ds \Rightarrow \dfrac{dt}{2} = (s+3)\,ds$

$\therefore ① \Rightarrow F(s) = \int \dfrac{dt/2}{t^2} \Rightarrow \dfrac{1}{2}\int t^{-2}\,dt \Rightarrow \dfrac{1}{2}\left[-\dfrac{1}{t}\right] \Rightarrow \dfrac{-1}{2(s^2+6s+13)}$

$\therefore L^{-1}[F'(s)] = -t\,L^{-1}[F(s)]$

$= -t\,L^{-1}\left[\dfrac{-1}{2(s^2+6s+13)}\right] \Rightarrow +\dfrac{t}{2}\,L^{-1}\left[\dfrac{1}{(s+3)^2+2^2}\right]$

$\Rightarrow +\dfrac{t}{2}e^{-3t}\,L^{-1}\left[\dfrac{1}{s^2+2^2}\right]$

$\Rightarrow \dfrac{t}{2}e^{-3t}L^{-1}\left[\dfrac{1}{s^2+2^2}\right] \Rightarrow \dfrac{te^{-3t}}{2}\,\dfrac{\sin 2t}{2} \Rightarrow \dfrac{te^{-3t}\sin 2t}{4}$

(2). $L^{-1}\left[\frac{s}{(s^2-a^2)^2}\right]$

Sol: $F'(s) = \frac{s}{(s^2-a^2)^2}$

Integrating w.r.to 's'

Put $s^2-a^2=t$

$2s\,ds = dt$

$s\,ds = dt/2$

$$\int F'(s)\,ds = \int \frac{s}{(s^2-a^2)^2}\,ds$$

$$F(s) = \int \frac{s}{(s^2-a^2)^2}\,ds \quad -(1)$$

$$(1) \Rightarrow F(s) = \int \frac{dt/2}{t^2} \Rightarrow \frac{1}{2}\int t^{-2}\,dt \Rightarrow \frac{1}{2}\left[-\frac{1}{t}\right]$$

$$\therefore F(s) = -\frac{1}{2t} = -\frac{1}{2(s^2-a^2)}$$

$$L^{-1}[F'(s)] = -t\,L^{-1}[F(s)]$$

$$= -t\,L^{-1}\left[-\frac{1}{2(s^2-a^2)}\right] \Rightarrow \frac{t}{2}L^{-1}\left[\frac{1}{s^2-a^2}\right]$$

$$\Rightarrow \frac{t}{2}\left[\frac{\sinh at}{a}\right] \Rightarrow \frac{t\sinh at}{2a}.$$

H.W

(3) $L^{-1}\left[\frac{s+2}{(s^2+4s+5)^2}\right]$

Ans: $\frac{te^{-2t}\sin t}{2}$

SPECIAL FUNCTIONS: {log, cot, tan}

$$\boxed{L^{-1}[F(s)] = -\frac{1}{t}L^{-1}[F'(s)]}$$

(1) $L^{-1}\left[\log\frac{s(s+1)}{s^2+1}\right]$

Sol: Let $F(s) = \log\frac{s(s+1)}{s^2+1}$

$$= \log s(s+1) - \log(s^2+1)$$

$$F(s) = \log s + \log(s+1) - \log(s^2+1)$$

- $\log\left(\frac{A}{B}\right) = \log A - \log B$
- $\log AB = \log A + \log B$

Diff w.r to 's'

$$F'(s) = \frac{1}{s} + \frac{1}{s+1} - \frac{1}{s^2+1}(2s)$$

$$\therefore L^{-1}[F(s)] = -\frac{1}{t} L^{-1}[F'(s)]$$

$$= -\frac{1}{t}\left\{L^{-1}\left[\frac{1}{s} + \frac{1}{s+1} - \frac{2s}{s^2+1}\right]\right\}$$

$$= -\frac{1}{t}\left\{L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s+1}\right] - 2L^{-1}\left[\frac{s}{s^2+1}\right]\right\}$$

$$= -\frac{1}{t}\left\{1 + e^{-t} - 2\cos t\right\}$$

2. $L^{-1}\left[\log\left(\frac{s^2+4}{(s-2)^2}\right)\right]$

Sol: $L^{-1}[F(s)] = -\frac{1}{t} L^{-1}[F'(s)]$

$$F(s) = \log\left(\frac{s^2+4}{(s-2)^2}\right)$$

$$F(s) = \log(s^2+4) - \log(s-2)^2 \Rightarrow \log(s^2+4) - 2\log(s-2)$$

$(\because \log a^m = m\log a)$

Diff w.r.to s

$$F'(s) = \frac{1}{s^2+4}(2s) - \frac{2}{s-2}$$

$$\therefore L^{-1}[F(s)] = -\frac{1}{t}\left\{L^{-1}\left[\frac{2s}{s^2+4} - \frac{2}{s-2}\right]\right\}$$

$$= -\frac{1}{t}\left\{2L^{-1}\left[\frac{s}{s^2+4}\right] - 2L^{-1}\left[\frac{1}{s-2}\right]\right\}$$

$$= -\frac{1}{t}\left\{2\cos 2t - 2e^{2t}\right\} \Rightarrow -\frac{2\cos 2t}{t} + \frac{2e^{2t}}{t}.$$

③ $L^{-1}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$

Sol: $F(s) = \log\left(\frac{s^2+a^2}{s^2+b^2}\right) \Rightarrow \log(s^2+a^2) - \log(s^2+b^2)$

Diff w.r to 's'

$$F'(s) = \frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}$$

$$L^{-1}[F(s)] = -\frac{1}{t} L^{-1}[F'(s)]$$

$$= -\frac{1}{t}\left\{L^{-1}\left[\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right]\right\}$$

$$= -\frac{1}{t}\left\{ L^{-1}\left[\frac{2s}{s^2+a^2}\right] - L^{-1}\left[\frac{2s}{s^2+b^2}\right]\right\}$$

$$= -\frac{1}{t}\left\{2\cos at - 2\cos bt\right\} \Rightarrow \frac{2\cos bt - 2\cos at}{t}.$$

④ $L^{-1}\left[\tan^{-1}\frac{2}{s}\right]$

Sol: $F(s) = \tan^{-1}\frac{2}{s}$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

Diff w.r to 's'

$$F'(s) = \frac{1}{1+\left(\frac{2}{s}\right)^2}\left(-\frac{2}{s^2}\right)$$

$$F'(s) = \frac{1}{\frac{s^2+2^2}{s^2}}\left(-2/s^2\right) \Rightarrow F'(s) = \frac{-2}{s^2+4}$$

$$\therefore L^{-1}[F(s)] = -\frac{1}{t}L^{-1}[F'(s)] \Rightarrow -\frac{1}{t}L^{-1}\left[-\frac{2}{s^2+4}\right]$$

$$\Rightarrow \frac{1}{t}L^{-1}\left[\frac{2}{s^2+4}\right] \Rightarrow \frac{\sin 2t}{t}.$$

⑤. $L^{-1}\left[\cot^{-1}\left(\frac{2}{s+1}\right)\right]$

$$\frac{d}{dt}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

Sol: $F(s) = \cot^{-1}\left(\frac{2}{s+1}\right)$

$$F'(s) = -\frac{1}{1+\frac{4}{(s+1)^2}}\left(-\frac{2}{(s+1)^2}\right)$$

$$F'(s) \Rightarrow \frac{2}{\frac{(s+1)^2+4}{(s+1)^2}}\left(\frac{1}{(s+1)^2}\right) \Rightarrow \frac{2}{(s+1)^2+4}$$

$$\therefore L^{-1}[F(s)] = -\frac{1}{t}L^{-1}[F'(s)] \Rightarrow -\frac{1}{t}\left\{L^{-1}\left[\frac{2}{(s+1)^2+4}\right]\right\}$$

$$\Rightarrow -\frac{1}{t}e^{-t}L^{-1}\left[\frac{2}{s^2+4}\right]$$

$$\Rightarrow -\frac{e^{-t}\sin 2t}{t}.$$

H.W

1. $L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right]$ Ans: $\frac{2\sinh t}{t}$

Method of Partial Fractions

① Find $L^{-1}\left[\dfrac{1-s}{(s+1)(s^2+4s+13)}\right]$

Sol: Consider,

$$\frac{1-s}{(s+1)(s^2+4s+13)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4s+13}$$

$$1-s = A(s^2+4s+13) + (Bs+C)(s+1)$$

Put $s=-1$

$$2 = A(1-4+13)+0$$

$$2 = 10A$$

$$\therefore \boxed{A = 1/5}$$

Put $s=0$

$$1 = 13A + C$$

$$1 = 13(1/5) + C$$

$$\therefore \boxed{C = -8/5}$$

Equating coefficient of s^2 on b/s, we get

$$0 = A + B \qquad \therefore \boxed{B = -1/5}$$

$$\frac{1-s}{(s+1)(s^2+4s+13)} = \frac{1/5}{s+1} + \frac{(-1/5)s + (-8/5)}{s^2+4s+13}$$

$$= \frac{1}{5}\left[\frac{1}{s+1}\right] - \frac{1}{5}\left[\frac{s+8}{s^2+4s+13}\right]$$

$$= \frac{1}{5}\left[\frac{1}{s+1}\right] - \frac{1}{5}\left[\frac{s+8}{(s+2)^2+9}\right]$$

$$= \frac{1}{5}\left[\frac{1}{s+1}\right] - \frac{1}{5}\left[\frac{s+2}{(s+2)^2+3^2} + \frac{6}{(s+2)^2+3^2}\right]$$

$$= \frac{1}{5}\left[\frac{1}{s+1}\right] - \frac{1}{5}\left[\frac{s+2}{(s+2)^2+3^2}\right] - \frac{6}{5}\left[\frac{1}{(s+2)^2+3^2}\right]$$

Apply Inverse Laplace on b/s:

$$L^{-1}\left[\frac{1-s}{(s+1)(s^2+4s+13)}\right] = \frac{1}{5}L^{-1}\left[\frac{1}{s+1}\right] - \frac{1}{5}L^{-1}\left[\frac{s+2}{(s+2)^2+3^2}\right] - \frac{6}{5}L^{-1}\left[\frac{1}{(s+2)^2+3^2}\right]$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}L^{-1}\left[\frac{s}{s^2+3^2}\right] - \frac{\cancel{6}2}{5}e^{-2t}L^{-1}\left[\frac{3}{s^2+3^2}\right]$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}\cos 3t - \frac{2}{5}e^{-2t}\sin 3t.$$

(2) Find $L^{-1}\left[\dfrac{5s^2-15s-11}{(s+1)(s-2)^3}\right]$

Sol: Consider $\dfrac{5s^2-15s-11}{(s+1)(s-2)^3} = \dfrac{A}{s+1} + \dfrac{B}{(s-2)} + \dfrac{C}{(s-2)^2} + \dfrac{D}{(s-2)^3}$

$5s^2-15s-11 = A(s-2)^3 + B(s+1)(s-2)^2 + C(s+1)(s-2) + D(s+1)$

Put $s=-1$, we get $5+15-11 = A(-1-2)^3$

$9 = -27A$ $\quad \therefore \boxed{A = -\tfrac{1}{3}}$

Equating coeff of s^3 on both sides, we get

$0 = A + B \Rightarrow B = -A \Rightarrow \boxed{B = \tfrac{1}{3}}$

Put $s=2$, We get $5(4)-30-11 = D(3) \Rightarrow -21 = 3D$ $\quad \boxed{\therefore D = -7}$

Put $s=0$, we get $-11 = -8A + 4B - 2C + D$

$= -8(-\tfrac{1}{3}) + 4(\tfrac{1}{3}) - 2C - 7$

$-4 = 8/3 + 4/3 - 2C \Rightarrow 4 - 2C = -8$

$\Rightarrow -2C = -8$ $\quad \boxed{\therefore C = 4}$

$\dfrac{5s^2-15s-11}{(s+1)(s-2)^3} = \dfrac{-1/3}{s+1} + \dfrac{1/3}{s-2} + \dfrac{4}{(s-2)^2} - \dfrac{7}{(s-2)^3}$

Apply Inverse Laplace on b/s:

$L^{-1}\left[\dfrac{5s^2-15s-11}{(s+1)(s-2)^3}\right] = -\tfrac{1}{3}L^{-1}\left[\dfrac{1}{s+1}\right] + \tfrac{1}{3}L^{-1}\left[\dfrac{1}{s-2}\right] + 4L^{-1}\left[\dfrac{1}{(s-2)^2}\right] - 7L^{-1}\left[\dfrac{1}{(s-2)^3}\right]$

$= -\tfrac{1}{3}e^{-t} + \tfrac{1}{3}e^{2t} + 4e^{2t}L^{-1}\left[\dfrac{1}{s^2}\right] - 7e^{2t}L^{-1}\left[\dfrac{1}{s^3}\right]$

$= -\tfrac{1}{3}e^{-t} + \tfrac{1}{3}e^{2t} + 4e^{2t}t - \tfrac{7}{2}e^{2t}L^{-1}\left[\tfrac{2}{s^3}\right]$

$\Rightarrow -\tfrac{1}{3}e^{-t} + \tfrac{1}{3}e^{2t} + 4te^{2t} - \tfrac{7}{2}e^{2t}t^2.$

3. H.W Find $L^{-1}\left[\dfrac{1}{s(s+1)(s+2)}\right]$ Ans: $\dfrac{1}{2}\left[1 + e^{-2t} - 2e^{-t}\right]$

CONVOLUTION THEOREM

Definition:-

The convolution of two functions $f(t)$ & $g(t)$ is defined as $f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$.

2 Marks:

Statement:- CONVOLUTION THEOREM.

If $f(t)$ & $g(t)$ are functions defined for $t \geq 0$,

then $L[f(t) * g(t)] = L[f(t)] \cdot L[g(t)]$

(or) $L^{-1}[F(s) \cdot G(s)] = L^{-1}[F(s)] * L^{-1}[G(s)]$

Note:- $f(t) * g(t) = g(t) * f(t)$

Problems:-

① Using Convolution theorem, find $L^{-1}\left[\dfrac{s}{(s^2+a^2)(s^2+b^2)}\right]$

Sol:-

$$L^{-1}[F(s) \cdot G(s)] = L^{-1}[F(s)] * L^{-1}(G(s))$$

$$L^{-1}\left[\frac{s}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{1}{s^2+b^2}\right]$$

$$= \cos at * \frac{\sin bt}{b}$$

$\left(\because f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du\right)$

$$= \int_0^t \cos au \, \frac{\sin b(t-u)}{b}\, du$$

$$= \frac{1}{b}\int_0^t \cos au \, \sin(bt-bu)\, du$$

$\left(\because \cos A \sin B = \dfrac{\sin(A+B) - \sin(A-B)}{2}\right)$

$$= \frac{1}{b}\int_0^t \left[\frac{\sin[(a-b)u+bt] - \sin[(a+b)u-bt]}{2}\right] du$$

$A = au$

$B = bt - bu$

$A+B = au+bt-bu = (a-b)u+bt$

$A-B = au-bt+bu = (a+b)u-bt$

$$\Rightarrow \frac{1}{2b}\int_0^t \left\{\left[\sin[(a-b)u+bt]\right] \cdot \sin[(a+b)u-bt]\right\} du$$

$$\Rightarrow \frac{1}{2b}\left\{ -\frac{\cos[(a-b)u+bt]}{(a-b)} + \frac{\cos[(a+b)u-bt]}{(a+b)}\right\}_{u=0}^{t}$$

$\cos(-\theta) = \cos\theta$

$$\Rightarrow \frac{1}{2b}\left\{\left(-\frac{\cos at}{a-b}+\frac{\cos at}{a+b}\right) - \left(-\frac{\cos bt}{a-b}+\frac{\cos bt}{a+b}\right)\right\}$$

Upper limit Lower limit

$$\Rightarrow \frac{1}{2b}\left\{-\frac{\cos at}{a-b}+\frac{\cos at}{a+b}+\frac{\cos bt}{a-b}-\frac{\cos bt}{a+b}\right\}$$

$$\Rightarrow \frac{1}{2b}\left\{\cos at\left(-\frac{1}{a-b}+\frac{1}{a+b}\right)+\cos bt\left(\frac{1}{a-b}-\frac{1}{a+b}\right)\right\}$$

$$\Rightarrow \frac{1}{2b}\left\{\cos at\left(\frac{-a-b+a-b}{(a-b)(a+b)}\right)+\cos bt\left(\frac{a+b-a+b}{(a-b)(a+b)}\right)\right\}$$

$$\Rightarrow \frac{1}{2b}\left\{\cos at\left(\frac{-2b}{a^2-b^2}\right)+\cos bt\left(\frac{2b}{a^2-b^2}\right)\right\}$$

$\div 2b$

$$\Rightarrow -\frac{\cos at}{a^2-b^2}+\frac{\cos bt}{a^2-b^2} \Rightarrow \frac{\cos bt - \cos at}{a^2-b^2}.$$

PRACTICE :-

1. Find $L^{-1}\left[\frac{s}{(s^2+1)(s^2+4)}\right]$

Replace $a=1$ $b=2$ (Refer above Problem)

Ans: $\frac{\cos t - \cos 2t}{3}$.

2. Find $L^{-1}\left[\frac{s}{(s^2+4)(s^2+9)}\right]$

Replace $a=2$ $b=3$ (Refer above Problem)

ANS: $\frac{\cos 2t - \cos 3t}{5}$

2. Using Convolution theorem find $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$

Sol:-

$$L^{-1}[F(s)\,G(s)] = L^{-1}[F(s)] * L^{-1}[G(s)]$$

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{1}{s^2+a^2}\right]$$

$$= \cos at * \frac{\sin at}{a}$$

$\left(\because \cos A \sin B = \frac{\sin(A+B) - \sin(A-B)}{2}\right)$

$A = au$
$B = at - au$
$A+B = au + at - au$
$A+B = at$
$A-B = au - at + au$
$= 2au - at$

$$= \frac{1}{a}\int_0^t \cos au \sin a(t-u)\,du$$

$$= \frac{1}{a}\int_0^t \left[\frac{\sin at - \sin(2au - at)}{2}\right] du$$

$$= \frac{1}{2a}\left[\sin at\,(u) + \frac{\cos(2au-at)}{2a}\right]_0^t$$

$(\because \cos(-at) = \cos at)$

$$= \frac{1}{2a}\left\{\left(\sin at\,(t) + \frac{\cos(at)}{2a}\right) - \left(0 + \frac{\cos at}{2a}\right)\right\}$$

$$= \frac{1}{2a}\left\{t\sin at + \frac{\cos at}{2a} - \frac{\cos at}{2a}\right\} \Rightarrow \frac{t\sin at}{2a}.$$

3. Find $L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$ using Convolution theorem.

Sol:-

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\left(\frac{s}{s^2+a^2}\right)\left(\frac{s}{s^2+b^2}\right)\right]$$

$$= L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{s}{s^2+b^2}\right]$$

$$= \cos at * \cos bt$$

$\left(\because \cos A \cos B = \frac{\cos(A+B) + \cos(A-B)}{2}\right)$

$$= \int_0^t \cos au \cos b(t-u)\,du$$

$$= \int_0^t \cos au \cos(bt - bu)\,du$$

$$\Rightarrow \frac{1}{2}\int_0^t \left[\cos(au+bt-bu)+\cos(au-bt+bu)\right]du$$

$A = au$
$B = bt - bu$
$A+B = au+bt-bu$
$A-B = au-bt+bu$

$$\Rightarrow \frac{1}{2}\left[\frac{\sin(au+bt-bu)}{a-b}+\frac{\sin(au-bt+bu)}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left\{\underbrace{\left[\frac{\sin(at+bt-bt)}{a-b}+\frac{\sin(at-bt+bt)}{a+b}\right]}_{\text{upper}}-\underbrace{\left[\frac{\sin bt}{a-b}+\frac{\sin(-bt)}{a+b}\right]}_{\text{Lower}}\right\}$$

$(\because \sin(-bt) = -\sin(bt))$

$$= \frac{1}{2}\left\{\frac{\sin at}{a-b}+\frac{\sin at}{a+b}-\frac{\sin bt}{a-b}+\frac{\sin bt}{a+b}\right\}$$

$$= \frac{1}{2}\left\{\sin at\left(\frac{1}{a-b}+\frac{1}{a+b}\right)+\sin bt\left(\frac{-1}{a-b}+\frac{1}{a+b}\right)\right\}$$

$$= \frac{1}{2}\left\{\sin at\left(\frac{a+b+a-b}{(a+b)(a-b)}\right)+\sin bt\left(\frac{-a-b+a-b}{(a-b)(a+b)}\right)\right\}$$

$$= \frac{1}{2}\left\{\sin at\left(\frac{2a}{a^2-b^2}\right)+\sin bt\left(\frac{-2b}{a^2-b^2}\right)\right\}$$

$$\div 2 \Rightarrow \frac{a\sin at - b\sin bt}{a^2-b^2}.$$

(4) Find $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right]$ using Convolution theorem.

Sol:-

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+a^2}\right]$$

$$= L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{s}{s^2+a^2}\right]$$

$$= \cos at * \cos at$$

$$= \int_0^t \cos au \cos a(t-u)\,du = \int_0^t \cos au \cos(at-au)\,du$$

$$= \frac{1}{2}\int_0^t \left[\cos(au+at-au) + \cos(au-at+au)\right] du$$

$$= \frac{1}{2}\int_0^t \left[\cos at + \cos(2au-at)\right] du$$

$\left(\because \cos A \cos B = \frac{\cos(A+B)+\cos(A-B)}{2}\right.$

$$= \frac{1}{2}\left[\cos at\,(u) + \frac{\sin(2au-at)}{2a}\right]_{u=0}^{u=t}$$

$(\because \sin(-\theta) = -\sin\theta$

$$= \frac{1}{2}\left\{\underbrace{\left(\cos at\,(t) + \frac{\sin at}{2a}\right)}_{\text{upper}} - \underbrace{\left(0 + \left(-\frac{\sin at}{2a}\right)\right)}_{\text{lower}}\right\}$$

$$= \frac{1}{2}\left\{t\cos at + \frac{\sin at}{2a} + \frac{\sin at}{2a}\right\}$$

$$= \frac{1}{2}\left\{t\cos at + \frac{2\sin at}{2a}\right\} \Rightarrow \frac{1}{2}\left\{t\cos at + \frac{\sin at}{a}\right\}$$

$$\Rightarrow \frac{1}{2a}\left\{at\cos at + \sin at\right\}$$

H.W

1. Find $L^{-1}\left[\frac{s^2}{(s^2+4)^2}\right]$

Refer above Problem

$a = 2$

Ans: $\frac{1}{4}\left[\sin 2t + 2t\cos 2t\right]$

5. Using Convolution theorem, Find $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$

Sol:- $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{1}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right]$

$$= L^{-1}\left(\frac{1}{s^2+a^2}\right) * L^{-1}\left[\frac{1}{s^2+a^2}\right]$$

$$= \frac{\sin at}{a} * \frac{\sin at}{a}$$

$$= \int_0^t \frac{\sin au}{a}\,\frac{\sin a(t-u)}{a}\,du$$

$$= \frac{1}{a^2}\int_0^t \sin au\,\sin(at-au)\,du$$

$$= \frac{1}{2a^2}\int_0^t \left[\cos(2au-at) - \cos(at)\right] du$$

$\because$ $\sin(A)\sin B = \dfrac{\cos(A-B)-\cos(A+B)}{2}$

$A = au$
$B = at - au$
$A+B = au + at - au$
$A - B = au - at + au$
$A-B = 2au - at$

$\sin(-\theta) = -\sin\theta.$

$$= \frac{1}{2a^2}\left[\frac{\sin(2au-at)}{2a} - \cos(at)\, u\right]_{u=0}^{u=t}$$

$$= \frac{1}{2a^2}\left\{\underbrace{\left(\frac{\sin at}{2a} - t\cos at\right)}_{\text{Upper}} - \underbrace{\left(-\frac{\sin at}{2a} - 0\right)}_{\text{Lower}}\right\}$$

$$= \frac{1}{2a^2}\left\{\frac{\sin at}{2a} - t\cos at + \frac{\sin at}{2a}\right\}$$

$$= \frac{1}{2a^2}\left\{\frac{2\sin at}{2a} - t\cos at\right\}$$

$$\Rightarrow \frac{1}{2a^2}\left\{\frac{\sin at}{a} - t\cos at\right\} \Rightarrow \frac{1}{2a^3}\left\{\sin at - at\cos at\right\}.$$

(6) Using Convolution theorem, find $L^{-1}\left[\dfrac{1}{s^2(s+5)}\right]$

Sol:- $L^{-1}\left[\dfrac{1}{s^2(s+5)}\right] = L^{-1}\left[\dfrac{1}{s^2}\right] * L^{-1}\left[\dfrac{1}{s+5}\right]$

$$= t * e^{-5t}$$

$$= \int_0^t u\, e^{-5(t-u)} du \Rightarrow \int_0^t u\, e^{-5t+5u} du$$

$$\Rightarrow \int_0^t u\, e^{-5t} e^{5u} du$$

$\int u\,dv = uv - u'v_1 + u''v_2 - \cdots$

u — I LATE

$$= e^{-5t}\int_0^t u\, e^{5u} du$$

$\downarrow$ Apply Bernoullis

$(\because e^0 = 1)$

$$= e^{-5t}\left[u\frac{e^{5u}}{5} - (1)\frac{e^{5u}}{25}\right]_0^t$$

$$= e^{-5t}\left[\left(\frac{te^{5t}}{5} - \frac{e^{5t}}{25}\right) - \left(0 - \frac{1}{25}\right)\right]$$

$$= e^{-5t}\left[\frac{te^{5t}}{5} - \frac{e^{5t}}{25} + \frac{1}{25}\right] \Rightarrow \frac{1}{25}\left[e^{-5t} + 5t - 1\right].$$

7) Find $L^{-1}\left[\frac{2}{(s+1)(s^2+4)}\right]$ Using Convolution theorem.

Sol:-

$$L^{-1}\left[\frac{2}{(s+1)(s^2+4)}\right] = L^{-1}\left[\frac{1}{s+1}\cdot\frac{2}{s^2+4}\right]$$

$$= L^{-1}\left[\frac{1}{s+1}\right] * L^{-1}\left[\frac{2}{s^2+4}\right]$$

W.K.T

$(f(t) * g(t)) = g(t) * f(t)$

$$= e^{-t} * \sin 2t$$

$$= \sin 2t * e^{-t}$$

$$= \int_0^t \sin 2u \, e^{-(t-u)} du$$

(∵ Formula:- $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$

$a=1 \quad b=2$

$$= \int_0^t \sin 2u \, e^{-t+u} du$$

$$= \int_0^t \sin 2u \, e^{-t} e^{u} du$$

$$= e^{-t}\int_0^t \sin 2u \, e^{u} du$$

↓ Apply $\int e^{ax}\sin bx\,dx$

$$= e^{-t}\left[\frac{e^u}{1^2+2^2}(\sin 2u - 2\cos 2u)\right]_{u=0}^{u=t}$$

$$= e^{-t}\left[\frac{e^u}{5}(\sin 2u - 2\cos 2u)\right]_{u=0}^{u=t}$$

$$= e^{-t}\left\{\underbrace{\frac{e^t}{5}(\sin 2t - 2\cos 2t)}_{\text{upper}} - \underbrace{\frac{e^0}{5}[\sin 0 - 2\cos 0]}_{\text{Lower}}\right\}$$

(∵ $e^0 = 1$, $\sin 0 = 0$, $\cos 0 = 1$)

$$= e^{-t}\left\{\frac{e^t}{5}(\sin 2t - 2\cos 2t) - \frac{1}{5}[0-2]\right\}$$

$$= \frac{e^{-t}e^{t}}{5}(\sin 2t - 2\cos 2t) + \frac{2e^{-t}}{5}$$

$$= \frac{1}{5}(\sin 2t - 2\cos 2t) + \frac{2e^{-t}}{5} \,//.$$

8) Find $L^{-1}\left[\frac{4}{(s^2+2s+5)^2}\right]$ using Convolution theorem.

Sol:-

$$L^{-1}\left[\frac{4}{(s^2+2s+5)^2}\right] = L^{-1}\left[\frac{2}{s^2+2s+5}\right] * L^{-1}\left[\frac{2}{s^2+2s+5}\right]$$

$$= L^{-1}\left[\frac{2}{(s+1)^2+4}\right] * L^{-1}\left[\frac{2}{(s+1)^2+4}\right]$$

$$= e^{-t} L^{-1}\left[\frac{2}{s^2+2^2}\right] * e^{-t} L^{-1}\left[\frac{2}{s^2+2^2}\right]$$

$$= e^{-t}\sin 2t * e^{-t}\sin 2t$$

$$= \int_0^t e^{-u}\sin 2u \; e^{-(t-u)}\sin 2(t-u)\,du$$

$$= \int_0^t e^{-u}\sin 2u \; e^{-t} e^{u} \sin(2t-2u)\,du$$

$$= e^{-t}\int_0^t \sin 2u \sin(2t-2u)\,du$$

$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

$A = 2u$
$B = 2t-2u$
$A+B = 2u+2t-2u$
$= 2t$
$A-B = 2u-2t+2u$
$= 4u-2t$

$$= e^{-t}\int_0^t \left[\frac{\cos(4u-2t) - \cos 2t}{2}\right] du$$

$$= \frac{e^{-t}}{2}\left\{\int_0^t \left(\cos(4u-2t) - \cos 2t\right) du\right\}$$

$$= \frac{e^{-t}}{2}\left[\frac{\sin(4u-2t)}{4} - \cos 2t \cdot u\right]_{u=0}^{t}$$

$\because \sin(-2t) = -\sin 2t$

$$= \frac{e^{-t}}{2}\left\{\underbrace{\left(\frac{\sin 2t}{4} - t\cos 2t\right)}_{\text{upper}} - \underbrace{\left(-\frac{\sin 2t}{4} - 0\right)}_{\text{lower}}\right]$$

$$= \frac{e^{-t}}{2}\left\{\frac{\sin 2t}{4} - t\cos 2t + \frac{\sin 2t}{4}\right\} \Rightarrow \frac{e^{-t}}{2}\left[\frac{\cancel{2}\sin 2t}{\cancel{4}\,2} - t\cos 2t\right]$$

$$\Rightarrow \frac{e^{-t}}{4}\left[\sin 2t - 2t\cos 2t\right]$$

(9) Find $L^{-1}\left[\frac{s+2}{(s^2+4s+13)^2}\right]$ using Convolution theorem.

Sol:-

$$L^{-1}\left[\frac{s+2}{s^2+4s+13}\cdot\frac{1}{s^2+4s+13}\right]$$

$$= L^{-1}\left(\frac{s+2}{s^2+4s+13}\right) * L^{-1}\left[\frac{1}{s^2+4s+13}\right]$$

$$= L^{-1}\left[\frac{s+2}{(s+2)^2+13-4}\right] * L^{-1}\left[\frac{1}{(s+2)^2+13-4}\right]$$

$$= L^{-1}\left[\frac{s+2}{(s+2)^2+9}\right] * L^{-1}\left[\frac{1}{(s+2)^2+9}\right]$$

$$= e^{-2t} L^{-1}\left[\frac{s}{s^2+3^2}\right] * e^{-2t} L^{-1}\left[\frac{1}{s^2+3^2}\right]$$

$$= e^{-2t}\cos 3t * e^{-2t}\frac{\sin 3t}{3}$$

$$= \frac{1}{3}\int_0^t e^{-2u}\cos 3u \; e^{-2(t-u)}\sin 3(t-u)\,du$$

$$= \frac{1}{3}\int_0^t \cancel{e^{-2u}}\cos 3u \; e^{-2t}\cancel{e^{2u}}\sin(3t-3u)\,du$$

$$= \frac{e^{-2t}}{3}\int_0^t \cos 3u\,\sin(3t-3u)\,du$$

$\downarrow$ $\cos A\sin B = \frac{1}{2}\left[\sin(A+B)-\sin(A-B)\right]$

$A = 3u \quad B = 3t-3u$

$A+B = 3u+3t-3u = 3t$

$A-B = 3u-3t+3u = 6u-3t$

$$= \frac{e^{-2t}}{3}\int_0^t \left[\frac{\sin 3t - \sin(6u-3t)}{2}\right]du$$

$$= \frac{e^{-2t}}{6}\int_0^t \left[\sin 3t - \sin(6u-3t)\right]du$$

$$= \frac{e^{-2t}}{6}\left\{\sin 3t.u - \left(-\frac{\cos(6u-3t)}{6}\right)\right\}_{u=0}^{t}$$

$$= \frac{e^{-2t}}{6}\left\{\sin 3t.u + \frac{\cos(6u-3t)}{6}\right\}_{u=0}^{t}$$

$$= \frac{e^{-2t}}{6}\left\{\left(t\sin 3t + \frac{\cos 3t}{6}\right) - \left(0 + \frac{\cos 3t}{6}\right)\right\}$$

Upper

$(\because \cos(-\theta) = \cos\theta)$

$$= \frac{e^{-2t}}{6}\left\{t\sin 3t + \cancel{\frac{\cos 3t}{6}} - \cancel{\frac{\cos 3t}{6}}\right\}$$

$$= \frac{e^{-2t}\, t\sin 3t}{6} \;//.$$

(10) Find $L^{-1}\left[\frac{s}{(s^2+2s+5)^2}\right]$ Using Convolution theorem.

Sol:-

$$L^{-1}\left[\frac{s}{(s^2+2s+5)^2}\right] = L^{-1}\left[\frac{s}{s^2+2s+5}\cdot\frac{1}{s^2+2s+5}\right]$$

$$= L^{-1}\left[\frac{s}{s^2+2s+5}\right] * L^{-1}\left[\frac{1}{s^2+2s+5}\right]$$

$$= L^{-1}\left[\frac{s}{(s+1)^2+4}\right] * L^{-1}\left(\frac{1}{(s+1)^2+4}\right)$$

$$= L^{-1}\left[\frac{(s+1)-1}{(s+1)^2+4}\right] * L^{-1}\left[\frac{1}{(s+1)^2+4}\right]$$

$$= L^{-1}\left[\frac{s+1}{(s+1)^2+4} - \frac{1}{(s+1)^2+4}\right] * L^{-1}\left[\frac{1}{(s+1)^2+4}\right]$$

$$= L^{-1}\left[\frac{s+1}{(s+1)^2+2^2} - \frac{1}{(s+1)^2+2^2}\right] * e^{-t} L^{-1}\left[\frac{1}{s^2+2^2}\right]$$

$$= e^{-t} L^{-1}\left[\frac{s}{s^2+2^2} - \frac{1}{s^2+2^2}\right] * e^{-t}\left(\frac{\sin 2t}{2}\right)$$

$$= e^{-t}\left[\cos 2t - \frac{\sin 2t}{2}\right] * \frac{e^{-t}}{2}\sin 2t$$

Using Convolution theorem,

$$= \int_0^t e^{-u}\left[\cos 2u - \frac{\sin 2u}{2}\right] \frac{e^{-(t-u)}}{2} \sin 2(t-u)\, du$$

$$= \int_0^t e^{-u}\left[\cos 2u - \frac{1}{2}\sin 2u\right] \frac{e^{-t}e^{u}}{2} \sin(2t-2u)\, du$$

$$= \frac{e^{-t}}{2}\int_0^t \left[\cos 2u - \frac{\sin 2u}{2}\right] \sin(2t-2u)\, du$$

$$= \frac{e^{-t}}{2}\int_0^t \left[\cos 2u \, \sin(2t-2u) - \frac{\sin 2u}{2} \sin(2t-2u)\right] du$$

$$\cos A \sin B = \frac{\sin(A+B) - \sin(A-B)}{2}$$

$A = 2u \quad B = 2t-2u$

$A+B = 2u+2t-2u = 2t$

$A-B = 2u-2t+2u = 4u-2t$

$$\sin A \sin B = \frac{\cos(A-B) - \cos(A+B)}{2}$$

$A = 2u \quad B = 2t-2u$

$A+B = 2u+2t-2u = 2t$

$A-B = 2u-2t+2u = 4u-2t$

$$= \frac{e^{-t}}{2}\int_0^t \left[\frac{\sin 2t - \sin(4u-2t)}{2} - \frac{1}{2}\left(\frac{\cos(4u-2t) - \cos 2t}{2}\right)\right] du$$

$$= \frac{e^{-t}}{4}\int_0^t \left[\sin 2t - \sin(4u-2t) - \frac{\cos(4u-2t)}{2} + \frac{\cos 2t}{2}\right] du$$

$$= \frac{e^{-t}}{4}\left\{\sin 2t . u + \frac{\cos(4u-2t)}{4} - \frac{\sin(4u-2t)}{2\times 4} + \frac{\cos 2t}{2} . u\right\}_{u=0}^{u=t}$$

$$= \frac{e^{-t}}{4}\left\{\left(\sin 2t . t + \frac{\cos 2t}{4} - \frac{\sin 2t}{8} + \frac{t\cos 2t}{2}\right) - \left(0 + \frac{\cos 2t}{4} + \frac{\sin 2t}{8} + 0\right)\right\}$$

$$= \frac{e^{-t}}{4}\left\{t \sin 2t + \cancel{\frac{\cos 2t}{4}} - \frac{\sin 2t}{8} + \frac{t\cos 2t}{2} - \cancel{\frac{\cos 2t}{4}} - \frac{\sin 2t}{8}\right\}$$

$$= \frac{e^{-t}}{4}\left\{t \sin 2t + \frac{t\cos 2t}{2} - \frac{2\sin 2t}{8}\right\}$$

$$= \frac{e^{-t}}{4}\left\{t\sin 2t + \frac{t\cos 2t}{2} - \frac{\sin 2t}{4}\right\}$$

$$= \frac{e^{-t}}{4}\left\{\frac{4t\sin 2t + 2t\cos 2t - \sin 2t}{4}\right\}$$

$$\Rightarrow \frac{e^{-t}}{16}\left\{4t\sin 2t + 2t\cos 2t - \sin 2t\right\}_{//}$$

H.W

1. $L^{-1}\left[\frac{1}{(s+a)(s+b)}\right]$ Ans: $\frac{e^{-bt} - e^{-at}}{a-b}$.

SOLVING A SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS BY LAPLACE TRANSFORM

Formula:-

* $L[y'(t)] = sL[y(t)] - y(0)$
* $L[y''(t)] = s^2L[y(t)] - sy(0) - y'(0)$
* $L[y'''(t)] = s^3L[y(t)] - s^2y(0) - sy'(0) - y''(0)$.

Problems:-

① Solve $y'' - 3y' + 2y = 1$ given that $y(0) = 0$, $y'(0) = 1$ by using laplace transform.

Sol: Given:- $y'' - 3y' + 2y = 1$

Take Laplace transforms on both Sides,

$$L[y''(t)] - 3L[y'(t)] + 2L[y(t)] = L[1]$$

$\left[s^2 L[y(t)] - s y(0) - y'(0)\right] - 3\left[s L[y(t)] - y(0)\right] + 2L[y(t)] = 1/s$

Also given $y(0)=0 \quad y'(0)=1$

$\therefore \left[s^2 L[y(t)] - 0 - 1\right] - 3\left[s L[y(t)] - 0\right] + 2L[y(t)] = \frac{1}{s}$

$s^2 L[y(t)] - 1 - 3s L[y(t)] + 2L[y(t)] = \frac{1}{s}$

$L[y(t)]\{s^2 - 3s + 2\} - 1 = 1/s$

$L[y(t)](s^2 - 3s + 2) = \frac{1}{s} + 1$

$L[y(t)](s^2 - 3s + 2) = \frac{s+1}{s}$

$s^2 - 3s + 2 = 0$
$\downarrow$ factorise using calci
$(s-1)(s-2)$

$\therefore L[y(t)] = \frac{s+1}{s(s^2-3s+2)}$

$\therefore L[y(t)] = \frac{s+1}{s(s-1)(s-2)}$

$\therefore y(t) = L^{-1}\left[\frac{s+1}{s(s-1)(s-2)}\right]$ — (*)

Resolving into Partial fractions

Consider, $\frac{s+1}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}$ — (1)

$\frac{s+1}{\cancel{s(s-1)(s-2)}} = \frac{A(s-1)(s-2) + Bs(s-2) + Cs(s-1)}{\cancel{s(s-1)(s-2)}}$

$\therefore \quad s+1 = A(s-1)(s-2) + Bs(s-2) + Cs(s-1)$

Put $s=0$ we get $0+1 = A(-1)(-2)$ $1 = 2A$ $\boxed{A = 1/2}$	Put $s=1$ $2 = B\,1(-1)$ $\boxed{B = -2}$	Put $s=2$ we get $3 = C(2)(1)$ $\boxed{C = 3/2}$

$$\therefore (1) \Rightarrow \frac{s+1}{s(s-1)(s-2)} = \frac{1/2}{s} + \frac{(-2)}{s-1} + \frac{(3/2)}{s-2}$$

Now, Apply Inverse Laplace on b/s:

$$L^{-1}\left[\frac{s+1}{s(s-1)(s-2)}\right] = \frac{1}{2}L^{-1}\left[\frac{1}{s}\right] - 2L^{-1}\left[\frac{1}{s-1}\right] + \frac{3}{2}L^{-1}\left[\frac{1}{s-2}\right]$$

$$= \frac{1}{2} - 2e^{t} + \frac{3}{2}e^{2t}$$

$$= \frac{1}{2}\left[1 - 4e^{t} + 3e^{2t}\right] //.$$

(2) using Laplace transform, Solve $y'' - 3y' + 2y = e^{-t}$ given that $y(0) = 1$, $y'(0) = 0$.

Sol:- Given $y'' - 3y' + 2y = e^{-t}$

Take Laplace transform on both sides,

$$L[y''(t)] - 3L[y'(t)] + 2L[y(t)] = L[e^{-t}]$$

$$[s^2 L[y(t)] - sy(0) - y'(0)] - 3[sL[y(t)] - y(0)] + 2L[y(t)] = \frac{1}{s+1}$$

Given:- $y(0) = 1$ $y'(0) = 0$

$$[s^2 L[y(t)] - s - 0] - 3[sL[y(t)] - 1] + 2L[y(t)] = \frac{1}{s+1}$$

$$s^2 L[y(t)] - s - 3sL[y(t)] + 3 + 2L[y(t)] = \frac{1}{s+1}$$

$$L[y(t)]\{s^2 - 3s + 2\} - s + 3 = \frac{1}{s+1}$$

$$L[y(t)](s^2 - 3s + 2) = \frac{1}{s+1} + s - 3$$

$$= \frac{1 + s^2 + s - 3s - 3}{s+1}$$

$$L[y(t)](s^2 - 3s + 2) = \frac{s^2 - 2s - 2}{s+1}$$

$$L[y(t)] = \frac{s^2-2s-2}{(s+1)(s^2-3s+2)}$$

$$L[y(t)] = \frac{s^2-2s-2}{(s+1)(s-1)(s-2)}$$

$$\therefore y(t) = L^{-1}\left[\frac{s^2-2s-2}{(s+1)(s-1)(s-2)}\right] \quad \text{—— } (*)$$

Now

Consider $\dfrac{s^2-2s-2}{(s+1)(s-1)(s-2)} = \dfrac{A}{s+1} + \dfrac{B}{s-1} + \dfrac{C}{s-2}$ —— (1)

$$\therefore s^2-2s-2 = A(s-1)(s-2) + B(s+1)(s-2) + C(s+1)(s-1)$$

Put $s=1$

$1-2-2 = B(2)(-1)$

$-3 = -2B$

$\boxed{\therefore B = 3/2}$

$s=2$

$4-4-2 = C(3)(1)$

$-2 = 3C$

$\boxed{C = -2/3}$

$s=-1$

$1+2-2 = A(-2)(-3)$

$1 = 6A$

$A = 1/6$

$$(1) \Rightarrow \frac{s^2-2s-2}{(s+1)(s-1)(s-2)} = \frac{1/6}{s+1} + \frac{3/2}{s-1} + \frac{-2/3}{s-2}$$

Apply Inverse Laplace on b/s

$$\therefore (*) \Rightarrow L^{-1}\left[\frac{s^2-2s-2}{(s+1)(s-1)(s-2)}\right] = \frac{1}{6}L^{-1}\left[\frac{1}{s+1}\right] + \frac{3}{2}L^{-1}\left[\frac{1}{s-1}\right] - \frac{2}{3}L^{-1}\left[\frac{1}{s-2}\right]$$

$$= \frac{1}{6}e^{-t} + \frac{3}{2}e^{t} - \frac{2}{3}e^{2t}.$$

(3) Solve using Laplace transform $x'' - 2x' + x = e^t$ when $x(0) = 2$, $x'(0) = 1$. (OR) $\dfrac{d^2x}{dt^2} - 2\dfrac{dx}{dt} + x = e^t$ with $x = 2$, $\dfrac{dx}{dt} = -1$ at $t=0$.

Sol:- Given: $x'' - 2x' + x = e^t$

Taking Laplace Transforms

$$L[x''(t)] - 2L[x'(t)] + L[x(t)] = L[e^t]$$

$$[s^2 L[x(t)] - s x(0) - x'(0)] - 2[sL[x(t)] - x(0)] + L[x(t)] = L[e^t]$$

Also given $x(0) = 2 \quad x'(0) = -1$

$$(s^2 L[x(t)] - 2s + 1) - 2[sL[x(t)] - 2] + L[x(t)] = \frac{1}{s-1}$$

$$s^2 L[x(t)] - 2s + 1 - 2sL[x(t)] + 4 + L[x(t)] = \frac{1}{s-1}$$

$$L[x(t)](s^2 - 2s + 1) - 2s + 5 = \frac{1}{s-1}$$

$$L[x(t)](s^2 - 2s + 1) = \frac{1}{s-1} + 2s - 5$$

$$L[x(t)](s-1)^2 = \frac{1}{s-1} + 2s - 2 - 3$$

$$L[x(t)](s-1)^2 = \frac{1}{s-1} + 2(s-1) - 3$$

$$\therefore L[x(t)] = \frac{1}{(s-1)^3} + \frac{2(s-1)}{(s-1)^2} - \frac{3}{(s-1)^2}$$

$$\therefore x(t) = L^{-1}\left[\frac{1}{(s-1)^3} + \frac{2}{(s-1)} - \frac{3}{(s-1)^2}\right]$$

$$= L^{-1}\left[\frac{1}{(s-1)^3}\right] + 2L^{-1}\left[\frac{1}{s-1}\right] - 3L^{-1}\left[\frac{1}{(s-1)^2}\right]$$

$$= e^t L^{-1}\left[\frac{1}{s^3}\right] + 2e^t L^{-1}\left[\frac{1}{s}\right] - 3e^t L^{-1}\left[\frac{1}{s^2}\right]$$

$$= e^t\left(\frac{t^2}{2!}\right) + 2e^t - 3e^t t.$$

$$= \frac{e^t}{2}[t^2 - 6t + 4).$$

(4) Using Laplace transform, solve

$y'' + y' = t^2 + 2t$ given $y = 4$, $y' = 2$ when $t = 0$

(OR)

(or) $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x$, $y = 4$ $\frac{dy}{dx} = -2$ when $x = 0$

Sol:- $y'' + y' = t^2 + 2t$

Take Laplace Transform on both sides,

$$L[y''(t)] + L[y'(t)] = L[t^2] + 2L[t]$$

$$\left[s^2 L[y(t)] - sy(0) - y'(0)\right] + \left[sL[y(t)] - y(0)\right] = \frac{2}{s^3} + \frac{2}{s^2}$$

Also given $y(0) = 4$, $y'(0) = -2$

$$s^2 L[y(t)] - 4s + 2 + \left[sL[y(t)] - 4\right] = \frac{2}{s^3} + \frac{2}{s^2}$$

$$s^2 L[y(t)] + sL[y(t)] - 4s - 2 = \frac{2+2s}{s^3}$$

$$L[y(t)](s^2+s) - 4s - 2 = \frac{2+2s}{s^3}$$

$$L[y(t)](s^2+s) = \frac{2+2s}{s^3} + 4s + 2$$

$$= \frac{2 + 2s + 4s^4 + 2s^3}{s^3}$$

$$L[y(t)] = \frac{4s^4 + 2s^3 + 2s + 2}{s^3(s^2+s)}$$

$$= \frac{4s^4 + 2s^3 + 2s + 2}{s^4(s+1)}$$

$$L[y(t)] = \frac{4s^4}{s^4(s+1)} + \frac{2s^3}{s^4(s+1)} + \frac{2s+2}{s^4(s+1)}$$

$$L[y(t)] = \frac{4}{s+1} + \frac{2}{s(s+1)} + \frac{2(s+1)}{s^4(s+1)}$$

$$= \frac{4}{s+1} + \frac{2}{s(s+1)} + \frac{2}{s^4}$$

$$y(t) = L^{-1}\left[\frac{4}{s+1}\right] + 2L^{-1}\left[\frac{1}{s(s+1)}\right] + 2L^{-1}\left[\frac{1}{s^4}\right]$$

$$= 4e^{-t} + 2L^{-1}\left[\frac{1}{s(s+1)}\right] + 2\left(\frac{t^3}{3!}\right) \quad — (*)$$

In ⊛, Consider $L^{-1}\left[\frac{1}{s(s+1)}\right]$

Partial fractions

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

$$1 = A(s+1) + Bs$$

Put $s=0 \Rightarrow \boxed{1=A}$

Put $s=-1 \Rightarrow \boxed{1=-B}$

$$\therefore \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

$$\therefore L^{-1}\left[\frac{1}{s(s+1)}\right] = L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right]$$

$$\boxed{L^{-1}\left[\frac{1}{s(s+1)}\right] = 1 - e^{-t}}$$

$$⊛ \Rightarrow y(t) = 4e^{-t} + 2(1-e^{-t}) + \frac{2t^3}{6}$$

$$= 4e^{-t} + 2 - 2e^{-t} + t^3/3$$

$$= t^3/3 + 2e^{-t} + 2 \text{ //.}$$

5. Solve $y'' - 3y' + 2y = 4t + e^{3t}$ where $y(0)=1$ $y'(0)=-1$ Using Laplace transform.

Sol:- $y'' - 3y' + 2y = 4t + e^{3t}$

Take Laplace transforms on b/s

$$L[y''(t)] - 3L[y'(t)] + 2L[y(t)] = 4L[t] + L[e^{3t}]$$

$$[s^2L[y(t)] - sy(0) - y'(0)] - 3[sL[y(t)] - y(0)] + 2L[y(t)] = 4L[t] + L[e^{3t}]$$

Given $y(0)=1$ $y'(0)=-1$

$$s^2L[y(t)] - s + 1 - 3[sL[y(t)] - 1] + 2L[y(t)] = \frac{4}{s^2} + \frac{1}{s-3}$$

$$s^2L[y(t)] - s + 1 - 3sL[y(t)] + 3 + 2L[y(t)] = \frac{4}{s^2} + \frac{1}{s-3}$$

$$(s^2 - 3s + 2)L[y(t)] - s + 4 = \frac{4}{s^2} + \frac{1}{s-3}$$

$$L[y(t)](s^2 - 3s + 2) - s + 4 = \frac{4(s-3) + s^2}{s^2(s-3)}$$

$$L[y(t)]\,(s^2-3s+2) = \frac{4(s-3)+s^2}{s^2(s-3)} + s - 4$$

$$= \frac{4s-12+s^2+s(s^2(s-3)) - 4s^2(s-3)}{s^2(s-3)}$$

$$L[y(t)]\,[(s-2)(s-1)] = \frac{4s-12+s^2+s^3(s-3)-4s^2(s-3)}{s^2(s-3)}$$

$$L[y(t)] = \frac{4s-12+s^2+s^4-3s^3-4s^3+12s^2}{s^2(s-3)(s-2)(s-1)}$$

$$\therefore L[y(t)] = \frac{s^4-7s^3+13s^2+4s-12}{s^2(s-3)(s-2)(s-1)}$$

$$\therefore y(t) = L^{-1}\left[\frac{s^4-7s^3+13s^2+4s-12}{s^2(s-3)(s-2)(s-1)}\right] \quad \text{— } (*)$$

Consider

$$\frac{s^4-7s^3+13s^2+4s-12}{s^2(s-3)(s-2)(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-3} + \frac{D}{s-2} + \frac{E}{s-1} \quad \text{— (1)}$$

$$\therefore s^4-7s^3+13s^2+4s-12 = As(s-3)(s-2)(s-1) + B(s-3)(s-2)(s-1)$$
$$+ Cs^2(s-2)(s-1) + Ds^2(s-3)(s-1) + Es^2(s-3)(s-2)$$

Put $s=0$

$-12 = B(-3)(-2)(-1)$

$-12 = -6B$

$\boxed{\therefore B = 2}$

Put $s=3$, we get

$3^4 - 7(3^3) + 13(9) + 12 - 12 = C(9)(1)(2)$

$81 - 189 + 117 = 18C$

$9 = 18C$

$\boxed{\therefore C = 1/2}$

Put $s=2$, we get

$16 - 56 + 52 + 8 - 12$

$= D(4)(-1)(1)$

$8 = -4D$

$\boxed{D = -2}$

Put $s=1$, we get

$1 - 7 + 13 + 4 - 12 = E(-2)(-1)$

$-1 = 2E$ $\quad \boxed{E = -1/2}$

Equating Coeff of s^4 on b/s

$1 = A + C + D + E$

$1 = A + \frac{1}{2} - 2 - \frac{1}{2}$

$1 = A - 2$ $\quad \therefore \boxed{A = 3}$

$$\therefore (1) \Rightarrow \frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{s^2(s-3)(s-2)(s-1)} = \frac{3}{s} + \frac{2}{s^2} + \frac{1/2}{s-3} + \frac{(-2)}{s-2} + \frac{(-1/2)}{s-1}$$

Apply inverse Laplace transform

$$L^{-1}\left[\frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{s^2(s-3)(s-2)(s-1)}\right] = 3L^{-1}\left[\frac{1}{s}\right] + 2L^{-1}\left[\frac{1}{s^2}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{s-3}\right] - 2L^{-1}\left[\frac{1}{s-2}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{s-1}\right]$$

$$(*) \Rightarrow \therefore y(t) = 3 + 2t + \frac{e^{3t}}{2} - 2e^{2t} - \frac{e^t}{2}$$

(6) Solve $\frac{d^2y}{dt^2} + 4y = \sin 2t$, given $y(0) = 3$ & $y'(0) = 4$.

Sol:-

$y'' + 4y = \sin 2t$

Take Laplace transforms on b/s!

$L[y''(t)] + 4L[y(t)] = L[\sin 2t]$

$[s^2 L[y(t)] - s y(0) - y'(0)] + 4L[y(t)] = \frac{2}{s^2+4}$

Given:- $y(0) = 3$ $\quad y'(0) = 4$

$s^2 L[y(t)] - 3s - 4 + 4L[y(t)] = \frac{2}{s^2+4}$

$(s^2+4)\, L[y(t)] = \frac{2}{s^2+4} + 3s + 4$

$L[y(t)] = \frac{2}{(s^2+4)^2} + \frac{3s}{(s^2+4)} + \frac{4}{(s^2+4)}$

$y(t) = 2L^{-1}\left[\frac{1}{(s^2+4)^2}\right] + 3L^{-1}\left[\frac{s}{s^2+2^2}\right] + 4L^{-1}\left[\frac{1}{s^2+2^2}\right]$

$$= \frac{2}{8} L^{-1}\left[\frac{(s^2+2^2)-(s^2-2^2)}{(s^2+2^2)^2}\right] + 3\cos 2t + \frac{4}{2}\sin 2t$$

$$= \frac{1}{4}\left\{L^{-1}\left[\frac{1}{(s^2+2^2)}\right] - L^{-1}\left[\frac{s^2-2^2}{(s^2+2^2)^2}\right]\right\} + 3\cos 2t + 2\sin 2t$$

$$= \frac{1}{4}\left\{\frac{\sin 2t}{2} - t\cos 2t\right\} + 3\cos 2t + 2\sin 2t$$

$$= \frac{1}{8}\sin 2t - \frac{1}{4} t\cos 2t + 3\cos 2t + 2\sin 2t$$

$$y(t) = \left(\frac{1}{8}+2\right)\sin 2t \; \left(3-\frac{t}{4}\right)\cos 2t.$$

$$y(t) = \frac{17}{8}\sin 2t + \left(3-\frac{t}{4}\right)\cos 2t.$$

7) Solve by using L.T $(D^2+9)y = \cos 2t$, given that if $y(0)=1$ $y(\pi/2) = -1$.

{This is a boundary value Problem, since the value of y at two different Points $t=0, t=\pi/2$ are given}

Sol:- Given:- $(D^2+9)y = \cos 2t$

$$D^2y + 9y = \cos 2t$$

$$y'' + 9y = \cos 2t$$

Take Laplace Transform on b/s:

$$L[y''(t)] + 9L[y(t)] = L[\cos 2t]$$

$$\{s^2 L[y(t)] - s\,y(0) - y'(0)\} + 9L[y(t)] = \frac{s}{s^2+4}$$

Also given $y(0)=1$ ~~take~~ $y'(0)=K$

$$(s^2+9)L[y(t)] - s - K = \frac{s}{s^2+4}$$

$$(s^2+9)L[y(t)] = \frac{s}{s^2+4} + s + K$$

$$L[y(t)] = \frac{s}{(s^2+4)(s^2+9)} + \frac{s}{s^2+9} + \frac{K}{s^2+9}$$

$$y(t) = L^{-1}\left[\frac{s}{(s^2+4)(s^2+9)}\right] + L^{-1}\left[\frac{s}{s^2+9}\right] + L^{-1}\left[\frac{k}{s^2+9}\right]$$

$$y(t) = L^{-1}\left[\frac{s}{(s^2+4)(s^2+9)}\right] + \cos 3t + k\,\frac{\sin 3t}{3} \quad \text{--- (1)}$$

Now, Consider $\frac{s}{(s^2+4)(s^2+9)}$ (Apply Partial fractions)

$$\frac{s}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

$$s = (As+B)(s^2+9) + (Cs+D)(s^2+4)$$

$$s = As^3 + Bs^2 + 9As + 9B + Cs^3 + Ds^2 + 4Cs + 4D$$

Equating Coeff of s^3:

$0 = A + C$

$\Rightarrow \boxed{C = -A}$

Eq. Coeff of s^2:

$0 = B + D$

$\boxed{D = -B}$

Eq. coeff of s:

$1 = 9A + 4C$ --- (2)

Eq. coeff of s^0 (or) Constant term

$0 = 9B + 4D$ --- (3)

(2) $\Rightarrow$ $9A + 4C = 1$

$9A + 4(-A) = 1$

$5A = 1$

$\boxed{A = 1/5}$

(3) $\Rightarrow$ $9B + 4D = 0$

$9B + 4(-B) = 0$

$5B = 0$

$\boxed{B = 0} \Rightarrow \boxed{D = 0}$

$\Rightarrow C = -A$

$\boxed{C = -1/5}$

$$\therefore \frac{s}{(s^2+4)(s^2+9)} = \frac{\frac{1}{5}s + 0}{s^2+4} + \frac{(-\frac{1}{5})s + 0}{s^2+9}$$

$$\frac{s}{(s^2+4)(s^2+9)} = \frac{1}{5}\left[\frac{s}{s^2+4}\right] - \frac{1}{5}\left[\frac{s}{s^2+9}\right]$$

$$L^{-1}\left[\frac{s}{(s^2+4)(s^2+9)}\right] = \frac{1}{5}L^{-1}\left[\frac{s}{s^2+4}\right] - \frac{1}{5}L^{-1}\left[\frac{s}{s^2+9}\right]$$

$$= \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t$$

$$\therefore (1) \Rightarrow y(t) = \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t + \cos 3t + \frac{K\sin 3t}{3}$$

$$y(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{K\sin 3t}{3} \quad \text{---} \quad (*)$$

Also given $y(\pi/2) = -1$

Put $t = \pi/2$

$$y(\pi/2) = 1/5\cos 2(\pi/2) + \frac{4}{5}\cos(3\pi/2) + \frac{K}{3}\sin 3\pi/2$$

$$-1 = 1/5(-1) + \frac{4}{5}(0) + \frac{K}{3}(-1)$$

$$-1 = -1/5 - K/3 \quad \Rightarrow -1 + 1/5 = -K/3$$

$$\Rightarrow -4/5 = -K/3 \quad \Rightarrow \boxed{K = \frac{12}{5}}$$

$$(*) \Rightarrow y(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t.$$

(8) Solve $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = \sin t$, if $\frac{dy}{dt} = 0$ and $y = 2$ when $t = 0$ using Laplace transforms.

Sol:- $y''(t) + 4y'(t) + 4y(t) = \sin t$

Take Laplace Transforms

$$L[y''(t)] + 4L[y'(t)] + 4L[y(t)] = L[\sin t]$$

$$[s^2L[y(t)] - sy(0) - y'(0)] + 4[sL[y(t)] - y(0)] + 4L[y(t)] = \frac{1}{s^2+1}$$

Given: $y(0) = 2$ $\quad y'(0) = 0$

$$[s^2L[y(t)] - 2s - 0] + 4[sL[y(t)] - 2] + 4L[y(t)] = \frac{1}{s^2+1}$$

$$s^2L[y(t)] - 2s + 4sL[y(t)] - 8 + 4L[y(t)] = \frac{1}{s^2+1}$$

$$[s^2+4s+4]\,L[y(t)]-2s-8=\frac{1}{s^2+1}$$

$$(s+2)^2 L[y(t)] = \frac{1}{s^2+1}+2s+8$$

$$L[y(t)] = \frac{1}{(s^2+1)(s+2)^2}+\frac{2s+8}{(s+2)^2}$$

$$= \frac{1}{(s^2+1)(s+2)^2}+\frac{2s+4+4}{(s+2)^2}$$

$$= \frac{1}{(s^2+1)(s+2)^2}+\frac{2s+4}{(s+2)^2}+\frac{4}{(s+2)^2}$$

$$= \frac{1}{(s^2+1)(s+2)^2}+\frac{2(s+2)}{(s+2)^2}+\frac{4}{(s+2)^2}$$

$$L[y(t)] = \frac{1}{(s^2+1)(s+2)^2}+\frac{2}{(s+2)}+\frac{4}{(s+2)^2}$$

$$\therefore y(t) = L^{-1}\left[\frac{1}{(s^2+1)(s+2)^2}\right]+2L^{-1}\left[\frac{1}{s+2}\right]+4L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$y(t) = L^{-1}\left[\frac{1}{(s^2+1)(s+2)^2}\right]+2e^{-2t}+4e^{-2t}t. \quad \text{—— (1)}$$

Now, Consider

$$\frac{1}{(s^2+1)(s+2)^2} = \frac{As+B}{s^2+1}+\frac{C}{s+2}+\frac{D}{(s+2)^2} \quad \text{—— (*)}$$

$$1=(As+B)(s+2)^2+C(s^2+1)(s+2)+D(s^2+1)$$

Put $s=-2$, we get

$1 = D(4+1)$

$\boxed{D=1/5}$

Eq. coeff of s^3 on b/s:

$0=A+C$

$A=-C$ —— (2)

Eq. coeff of s^2 on b/s:

$0=4A+B+2C+D$

$0=4A+B-2A+D$

$0=2A+B+1/5$

$2A+B=-1/5$ —— (3)

Put $s=0$, we get

$$1 = 4B + 2C + D$$

$$1 = 4B - 2A + 1/5$$

$$-2A + 4B = 1 - 1/5$$

$$-2A + 4B = 4/5 \quad \text{--- (4)}$$

Solve (3) & (4)

$$\cancel{2A} + 4B = -1/5$$
$$-\cancel{2A} + 4B = 4/5$$
$$5B = 3/5$$

$$\boxed{\therefore B = 3/25}$$

Sub 'B' value in (3)

$$2A + \frac{3}{25} = -1/5$$

$$2A = -1/5 + 3/25$$

$$2A = -\frac{5}{5} + 3/25$$

$$2A = -\frac{2}{25} \Rightarrow \boxed{A = -\frac{4}{25}}$$

$$\Rightarrow C = -A$$

$$\boxed{C = 4/25}$$

$\therefore$ (*) $\Rightarrow$

$$\frac{1}{(s^2+1)(s+2)^2} = \frac{-\frac{4}{25}s + \frac{3}{25}}{s^2+1} + \frac{4/25}{s+2} + \frac{1/5}{(s+2)^2}$$

$$\frac{1}{(s^2+1)(s+2)^2} = -\frac{4}{25}\left(\frac{s}{s^2+1}\right) + \frac{3}{25}\left(\frac{1}{s^2+1}\right) + \frac{4}{25}\left[\frac{1}{s+2}\right] + \frac{1}{5}\left[\frac{1}{(s+2)^2}\right]$$

$$\therefore L^{-1}\left[\frac{1}{(s^2+1)(s+2)^2}\right] = -\frac{4}{25}L^{-1}\left[\frac{s}{s^2+1}\right] + \frac{3}{25}L^{-1}\left[\frac{1}{s^2+1}\right] + \frac{4}{25}L^{-1}\left[\frac{1}{s+2}\right] + \frac{1}{5}L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$= -\frac{4}{25}\cos t + \frac{3}{25}\sin t + \frac{4}{25}e^{-2t} + 1/5\, e^{-2t}t$$

$$\therefore (1) \Rightarrow y(t) = -\frac{4}{25}\cos t + \frac{3}{25}\sin t + \frac{4}{25}e^{-2t} + 1/5\, te^{-2t} + 2e^{-2t} + 4te^{-2t}$$

$$= -\frac{4}{25}\cos t + \frac{3}{25}\sin t + \frac{54}{25}e^{2t} + \frac{21}{5}te^{-2t}.$$

H.W

1. Solve $y'' - 4y' + 8y = e^t$ when $y(0)=2$, $y'(0)=1$
2. Solve $(D^2 - 3D + 2)y = e^{3t}$ with $y(0)=1$ and $y'(0)=0$

SECOND SHIFTING PROPERTY

$$L^{-1}\left[e^{-as}F(s)\right] = f(t-a)\,U(t-a)$$
$$= \left\{L^{-1}\left[F(s)\right]\right\}_{t\to t-a} U(t-a)$$

Problems:-

① $L^{-1}\left[\dfrac{e^{-\pi s}}{s+3}\right]$

Sol:- $L^{-1}\left[e^{-as}F(s)\right] = \left\{L^{-1}\left[F(s)\right]\right\}_{t\to t-a} U(t-a)$

$$L^{-1}\left[e^{-\pi s}\cdot\frac{1}{s+3}\right] = \left\{L^{-1}\left[\frac{1}{s+3}\right]\right\}_{t\to t-\pi} U(t-\pi)$$
$$= \left\{e^{-3t}\right\}_{t\to t-\pi} U(t-\pi)$$
$$= e^{-3(t-\pi)}\,U(t-\pi).$$

H.W

1. $L^{-1}\left[\dfrac{e^{-as}}{s}\right]$

Ans: $U(t-a)$

2. $L^{-1}\left[\dfrac{e^{-s}}{(s+1)^3}\right]$

Ans: $e^{-(t-1)}\dfrac{(t-1)^2}{2}\,U(t-1)$

② $L^{-1}\left[\dfrac{e^{-s}}{(s+1)(s+3)}\right]$

Sol:- $L^{-1}\left[e^{-s}\cdot\dfrac{1}{(s+1)(s+3)}\right] = \left\{L^{-1}\left[\dfrac{1}{(s+1)(s+3)}\right]\right\}_{t\to t-1} U(t-1)$

$$= \left\{L^{-1}\left[\frac{1/2}{s+1} - \frac{1/2}{s+3}\right]\right\}_{t\to t-1} U(t-1)$$
$$\Rightarrow \left\{\frac{1}{2}\left(e^{-t} - e^{-3t}\right)\right\}_{t\to t-1} U(t-1)$$
$$\Rightarrow \tfrac{1}{2}\left\{e^{-(t-1)} - e^{-3(t-1)}\right\} U(t-1).$$

$$\frac{1}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}$$
$$1 = A(s+3) + B(s+1)$$

$s=-1$ $\quad 1 = A(2) \;\therefore A = 1/2$

$s=-3$ $\quad 1 = -2B \;\therefore B = -1/2$

$$\frac{1}{(s+1)(s+3)} = \frac{1/2}{s+1} - \frac{1/2}{s+3}$$

Additional Problems:-

① Give an example of a function such that it has Laplace transformation exists even though it does not satisfy the sufficient conditions.

Sol:- $f(t) = \dfrac{1}{\sqrt{t}}$ (or) $t^{-1/2}$

It is not continuous at $t=0$, but $L\left[1/\sqrt{t}\right]$ exists.

$$L\left[\frac{1}{\sqrt{t}}\right] = L\left[t^{-1/2}\right] = \frac{\Gamma(-1/2+1)}{s^{-1/2+1}} \Rightarrow \frac{\Gamma(+1/2)}{s^{1/2}} \Rightarrow \frac{\sqrt{\pi}}{\sqrt{s}}.$$

TRANSFORMS OF UNIT STEP FUNCTION & UNIT IMPULSE FUNCTION

Def:- UNIT STEP FUNCTION

Unit step function is defined as $U(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$ where $a > 0$

Laplace Transform of unit step function:-

$$L[U(t-a)] = \frac{e^{-as}}{s}.$$

Proof:-

$$L[u(t-a)] = \int_0^\infty e^{-st}\, U(t-a)\, dt$$

$$= \int_0^a (0)\, dt + \int_a^\infty e^{-st} (1)\, dt \Rightarrow \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$= \left[0 - \left(\frac{e^{-sa}}{-s}\right)\right] \Rightarrow \frac{e^{-as}}{s}.$$

Def:- UNIT IMPULSE FUNCTION (OR) DIRAC DELTA FUNCTION

It is defined as $\delta(t-a) = \begin{cases} 0 & t \neq a \\ \infty & t = a \end{cases}$ such that

$$\int_{-\infty}^{\infty} \delta(t-a)\, dt = 1.$$

{∵ large force applied for a very short time.}

LAPLACE TRANSFORMS.

8 Mark topics.

I. Laplace Transforms of Periodic functions } → 1 Q.

II Convolution theorem → 1 Q

III Solution of ODE using L.T → 1 Q.

IV * I.V.T & F.V.T (Proof & Problems)
* Multiplication of t
* Division by t
* Improper integrals [type I, type II] (ie., $\int_0^\infty$ type, $\int_0^t$ type.
* Special function {$\cot^{-1}$, $\tan^{-1}$, log etc, for Inverse Laplace.
* Laplace Inverse of second shifting theorem

} → 1 Q (Two subdivisions)

2 Mark topics:-

* Sufficient Condition of L.T
* Define Laplace transform, Convolution thm, Unit step function, unit impulse function,
* Statement of I.V.T & F.V.T & its simple Problems.
* Is the linearity Property is applicable?
* Problems based on L.T & Inverse L.T
* Problems based on inverse second shifting thm,
* Special inverse function L.T [$\cot^{-1}$, $\tan^{-1}$, log]
* Change of scale Property.